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Article

Analysis of patterns in the harmonics of a string with artificially enforced nodes

Armaan Jain[1], Curtis Broadbent[2]
[1]Good Shepherd International School, Tamil Nadu, India
[2]Eisco Scientific North America, Victor, New York

SUMMARY

This study examines the higher harmonics in an oscillating string by analyzing the sound produced by a guitar with a spectrum analyzer. Higher harmonics of a string are simultaneously oscillating modes which have frequencies that are integer multiples of the fundamental frequency of the string. These harmonics can be viewed on an audio spectrum analyzer. They are always present in an oscillating string and contribute to its timbre and tonal quality. Specific higher harmonics can be produced directly by placing nodes (points where the string cannot oscillate) at different lengths along a string. The tone thus produced lacks the fundamental frequency but also has a very different harmonic structure. In a guitar string, for example, it is this harmonic structure which gives rise to the very different tonal quality of a plucked harmonic as compared to the directly excited pitch of the same frequency. We mathematically hypothesized that the higher harmonics in the series of the directly excited 2nd harmonic contain the alternate frequencies of the fundamental series, the higher harmonics of the directly excited 3rd harmonic series contain every third frequency of fundamental series, and so on. We also verify a simple mathematical relationship between two different harmonic series arising from two different boundary conditions that feature the same fundamental mode. To test our hypotheses, we enforced artificial nodes to excite the 2nd, 3rd, and 4th harmonics directly, and analyzed the resulting spectrum to verify the mathematical hypothesis. The data analysis corroborates both hypotheses.

INTRODUCTION

When we pluck a string in tension, a series of frequencies greater than the fundamental frequency is observed. Here the fundamental frequency is calculated using the string's wave speed and the string length. Theoretically, however, exciting the fundamental mode (the lowest or the "true" frequency) of a string in tension should not produce overtones (any higher frequency standing waves). In practice a range of overtones are produced, from which the non-resonant modes (frequencies that are not an integer multiple of the fundamental mode) decay quickly, while the fundamental mode and the harmonic overtones (frequencies that are part of the harmonic series i.e., are integer multiples of the fundamental mode) persist.

The modes whose frequencies are multiples of the fundamental frequency are called the higher harmonics. When we calculate the wave frequency using wave speed and wavelength, the result is only the fundamental frequency. However, in a string with closed ends, this is not a complete representation of the frequencies that can be supported. For the complete picture, the fundamental frequency can be used to find the higher harmonic frequencies. This is done by multiplying the fundamental mode by consecutive integers. This paper explores patterns existing in these higher harmonics of standing waves that can help us predict the resonant frequencies in a given system. This knowledge can be essential in many engineering applications from electrical power systems (1) to acoustics.

First, we derive a relationship between the frequency of the fundamental mode and its harmonics and then experimentally test that hypothesis. **Figure 1A** represents the fundamental mode of the open guitar string. The wavelength is twice the length of the oscillating string. **Figure 1B**, **1C** and **1D** show the 2nd, 3rd, and 4th harmonics of this string. The 2nd harmonic has a wavelength that is equal to half of the wavelength of the fundamental. The 3rd harmonic has a wavelength that is a third of the wavelength of the fundamental, and this pattern repeats for higher-order harmonics.

In this way, the wavelength of the n^{th} harmonic in the series is given by $\lambda_n = 1/n \times \lambda$. Here, λ is the wavelength of the fundamental excited mode. Using the general relationship between wavelength, frequency, and the constant velocity of the string wave $v = \lambda_n f_n$, we can calculate the frequency for all the higher harmonics on this string using Equation 1,

$$f_n = nv/\lambda = nf. \quad (1)$$

Using this method, we see that the frequencies of the harmonics are multiples of the fundamental frequency, as n is an integer. This is consistent with the idea that the frequency of higher harmonics of a string are the whole number multiples of the fundamental frequency.

If we examine the two nodes in **Figure 1A** we see they are placed such that every other wave has its nodes at these points. Therefore, all the three waves (part B, C and D) may

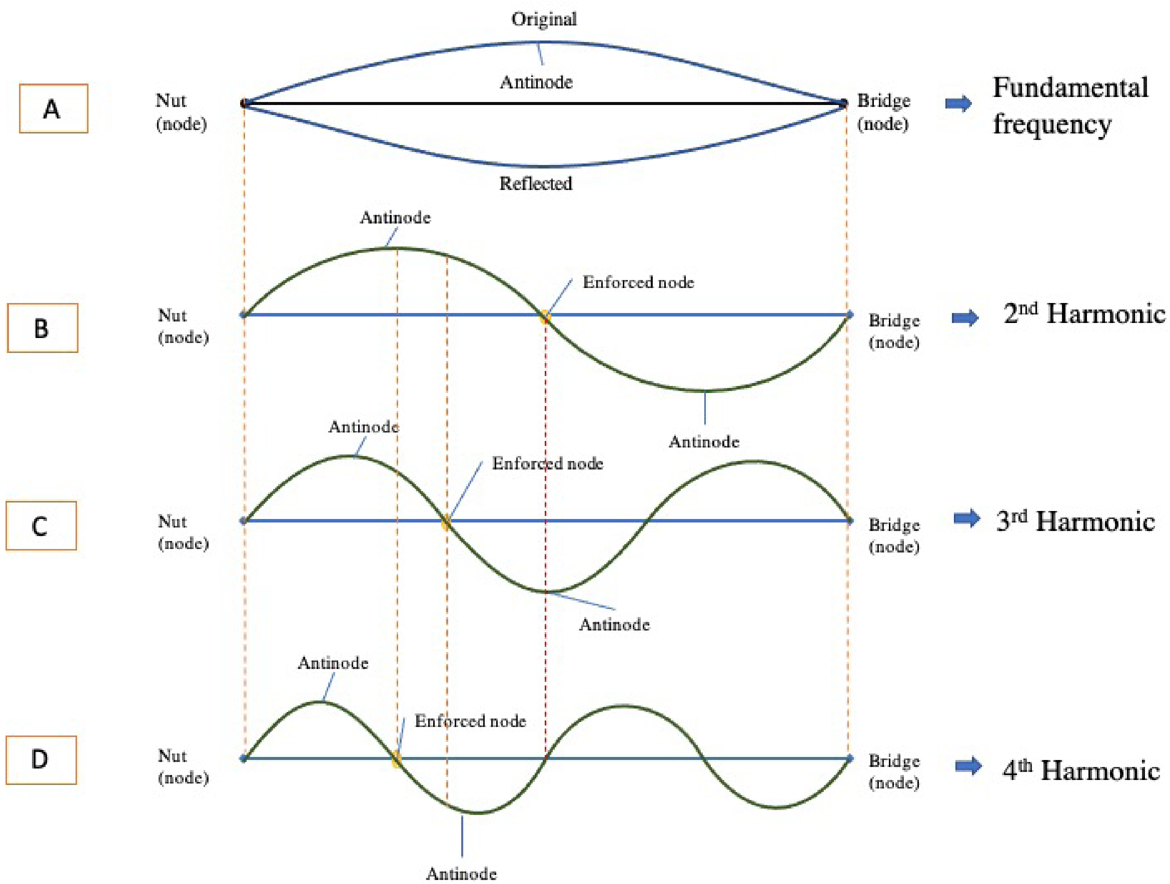

Figure 1. Schematic of the string of a guitar clamped on both ends (neck and bridge) without any artificially enforced nodes. These ends offer no lateral movement of the string, enforcing fixed nodes and providing the boundary conditions for the lowest frequency in the study. Part A, B, C and D are the 1st, 2nd , 3rd, and 4th harmonics respectively. The point labelled "enforced node" shows the point where the artificial node was enforced. This is half the length of the string from the neck for the 2nd harmonic, one third the length of the string for the 3rd harmonic, and one fourth the length of the string for the 4th harmonic. The dotted line is a projection of the nodes onto the other modes. When an enforced node of one mode lines up with a node of another mode, the modes can co-exist when the node is enforced. When this is not the case, the modes cannot co-exist when the node is enforced.

propagate without any destructive interference from nodes of the fundamental frequency in **Figure 1**.

Now, consider the 2nd and 3rd harmonics. In addition to appearing when the fundamental mode is excited, the 2nd and 3rd harmonics can be excited by enforcing a "node" at the locations given. However, enforcing an artificial node can cause problems for other modes in the system. The point labeled "enforced node" in the 2nd harmonic's graph is at the antinode of the 3rd harmonic (**Figure 1**). When the 2nd harmonic is produced, the artificial node does not permit the antinode to form at that point for the 3rd harmonic. Meanwhile, the node for the 4th harmonic is in line with the enforced node of the 2nd harmonic; thus, the 4th harmonic propagates without any restrictions from the enforced node of the 2nd harmonic.

In terms of integers, when we enforce the node for the 2nd harmonic, the 4th harmonic is also present in the series, while the 3rd harmonic is not. We can extrapolate this principle to higher order modes, as shown in **Table 1**, which shows that when the 2nd harmonic is excited using an enforced node, the harmonic frequencies and the alternate frequencies of the higher harmonics of the fundamental mode are the same.

We can derive these relationships mathematically from the fundamental properties of waves. Before doing so, it is important to define the terms that we will use. Modes refer to solutions of the wave equation on a string with specific boundary conditions. The fundamental mode of a guitar string is the lowest frequency mode of a string that is clamped at two ends. The higher-order modes are solutions to the wave equation with the same boundary conditions. When another node is enforced somewhere on the string, this additional boundary condition would result in a different set of solutions for the wave equation, even though they might have the same frequency as harmonics of the string with no enforced node. Therefore, in the following text, we will refer to harmonics of the excited mode as the series with the additional boundary condition to distinguish between these different sets of solutions.

Consider when an enforced node is used to excite the 3rd harmonic of the base mode with a frequency of 300 Hz. Integer multiples of the frequency would be 300 Hz, 600 Hz,

Table 1. Hypothetical harmonic series to explain the hypothesis.

Fundamental frequency 100 Hz	Fundamental frequency 200 Hz
100 Hz	
200 Hz	200 Hz
300 Hz	
400 Hz	400 Hz
500 Hz	
600 Hz	600 Hz
700 Hz	
800 Hz	800 Hz
900 Hz	
1000 Hz	1000 Hz

Note: Summary of data collected compared to the theoretically derived frequencies. Three sets of trials were conducted for each boundary condition, and each frequency value given in this table is the average of these three trials. The frequencies are the peak values discussed in ***Figure 3*** *and are the overtones for each boundary condition. The data has been placed alongside the theoretically predicted values in such a way that a frequency value in the empirical data that is close to or the same as the theoretically predicted value are in the same row.*

900 Hz, and so on. These are all 3rd multiples of the higher harmonic frequencies of the fundamental or base mode of 100 Hz. This can be generalized as follows: the relationship of the harmonics arising from different excited modes to a related fundamental mode can be found for two different boundary conditions, where the frequency of the fundamental mode from one set of boundary conditions is divisible by the frequency of the fundamental mode of another set of boundary conditions by a factor of n. In this case, the harmonic series with the higher fundamental frequency will contain harmonics which are every n^{th} multiple of the lower frequency.

For example, enforcing a node in the center of the string gives a fundamental excited mode with a frequency of 200 Hz. The second harmonic of the excited mode has a frequency of 400 Hz. With no enforced node, the fundamental excited frequency is 100 Hz, with the 2nd and 4th harmonics having frequencies of 200 Hz and 400 Hz, respectively. Thus, the fundamental frequency with the added boundary condition, which is twice the fundamental frequency in the original string, has a harmonic series comprising every second (or every alternate) frequency of the harmonic series of the original string.

We will now explore how the harmonics of the string with no enforced node are related to the harmonics of the same string with an enforced node. The relationship between the harmonics arising from different excited modes can be explained as follows: for any fundamental frequency, f, the frequency of the n^{th} harmonic of the m^{th} mode excited using an enforced node for the m^{th} mode is the same as the m^{th} harmonic of the n^{th} mode excited using an enforced node for the n^{th} node. A mathematical derivation is shown in Equations 2-4. For the fundamental or base mode, f_1, the frequency, f_n, of its n^{th} harmonic is

$$f_n = f_1 n. \qquad (2)$$

If we excite the n^{th} harmonic directly by using an artificially enforced node, then the frequency, f_{mn}, of the m^{th} harmonic of that n^{th} mode excited using an enforced node is

$$f_{mn} = f_n m. \qquad (3)$$

Now, substituting f_n in the equation we get

$$f_{mn} = f_1 nm. \qquad (4)$$

Notice that for the multiplication of f_1, the numbers n and m can be switched without making a difference to the product. This means that $f_{mn} = f_{nm}$. As a result, we may say, for example, that the frequency of the 2nd harmonic of the fundamental mode excited with an enforced node at the position of a node of the 3rd harmonic of the string's fundamental mode (with no enforced nodes) is the same as the 3rd harmonic of the fundamental mode excited with an enforced node at the position of a node of the 2nd harmonic of the string's fundamental mode. Through data analysis of the harmonic series produced using a guitar string, we found that both these theoretical patterns are valid in scenarios where the wave equation applies, and therefore can be used to predict the resonant frequencies in other systems.

RESULTS

We performed an experiment to validate the hypothesis derived above. The data was collected with a guitar and a phone-based frequency spectrum analyzer.

To verify the presence of the alternating pattern, we excited the harmonics directly, using the frets of a guitar as markers. Frets are metal bars on the guitar fingerboard used to control pitch. First, the high E string was tuned to a base mode of 330 Hz. The 2nd harmonic was produced by enforcing a node on the 12th fret, halfway through the string. This did not allow the string to oscillate at that point. The 3rd harmonic was produced by enforcing a node at the 9th fret, one-third of the string length. The 4th harmonic was produced by enforcing a node at the 7th fret, a quarter of the length of the string. The frequencies at which there were peaks in the graph were recorded in three separate trials. **Table 2** shows the average of these three trials to the nearest whole number. The theoretical values in **Table 2** were computed using the following calculation. The measured length of the string between the two ends was 0.648 m, and the speed of the wave is given by $v = \lambda f$. λ here was twice the distance between the two ends of the string 1.296 m. The wave speed was therefore equal to 427.68 ms^{-1}. The speed of a wave on a string is dependent only on the tension and linear density of the string (2). Since both of these parameters are constant, the wave speed is constant for all harmonic frequencies. We can then determine the wavelength of each wave by using the wave speed.

Figure 2 is a screenshot of the spectrum analyzer from the phone, showing a graph of frequency (Hz) on the x-axis

Figure 2. Graph of frequency against loudness for a standing wave as given by the spectrum analyzer app, showing how the data in Table 2 was collected. Each peak here is the overtone of the respective harmonic series, and we used a graph tracing feature to determine the frequency at these peaks. The 'peak hold' records initial graph as the sound fades away with time. This was used as a guiding graph to trace. The pause button allowed us to stop the application from collecting further samples, giving enough time to record the peak frequencies accurately.

Table 2. Summary of data collected compared to the theoretically derived frequencies.

Theoretical	Fundamental frequency peak frequency average	2nd Harmonic peak frequency average	3rd Harmonic peak frequency average	4th harmonic peak frequency average
330	328			
660	658	659		
990	990		989	
1320	1318	1318		1321
1650	1647			
1980	1976	1977	1979	
2310	2303			
2640	2633	2639		2638
2970	2964		2970	
3300	3296	3300		
3630	3625			
3960	3964	3961	3958	3963
4290	4285			
4620	4620	4623		
4950	4959		4963	
5280	5283	5282		5286
5610	5613			
5940	5947	5948	5948	
6270	6270			
6600	6602	6610		
6930	6931		6949	
7260	7258	7263		
7590	7608			
7920	7926	7939		
8250	8269			
8580	8589	8597		
8910	8937			
9240	9253	9275		
9570	9608			

*Note: Three sets of trials were conducted for each boundary condition, and each frequency value given in this table is the average of these three trials. The frequencies are the peak values discussed in **Figure 2** and are the overtones for each boundary condition. The data has been placed alongside the theoretically predicted values so that a frequency value in the empirical data that is close to or the same as the theoretically predicted value is in the same row.*

against loudness (dB) on the y-axis. A tracking feature was used to manually find the loudest frequency, corresponding to the peak. The frequency at these 'peaks' were recorded.

The spectrum in **Figure 2** has peaks at frequencies greater than 330 Hz. This verifies the presence of higher harmonics in a vibrating string. **Table 2** summarizes the data collected (loudest frequencies), including the theoretical values and the average values for fundamental, 2nd, 3rd and 4th harmonic series, placed such that the same frequency is in the same row. The empirical frequencies were entered into the chart in the row that had the closest theoretical harmonic frequency.

With this approach, we can see that every other box in the 2nd harmonic column is filled, every third box in the 3rd harmonic column is filled, and every fourth box in the 4th harmonic column is filled. This agrees with the results of our theoretical analysis. Moreover, if we consider the common frequencies, the frequency of the 3rd harmonic (row 6) in the 2nd harmonic peak frequency column is the same value as the 2nd harmonic in the 3rd harmonic peak frequency column. This example corroborates the hypothesis that $f_{mn} = f_{nm}$.

DISCUSSION

Our data analysis verifies the presence of higher harmonics generated by a plucked string. It also shows that there is a pattern in different harmonic series, provided they stem from the same fundamental series.

The deviation from the theoretical values increased at higher frequencies because the amplitude of the harmonics was much lower. As a result, background noise lead to measurement errors for these frequencies. To avoid this error, the background noise spectrum was initially noted, and once the spectra approached the loudness of the background noise, the peak frequencies were not recorded. Since the plot was analyzed for the peaks over time, the background noise was too soft to affect the peaks at lower frequencies. Furthermore, the limitations posed by the phone's microphone and operating system (OS) would enhance this uncertainty. Most smartphone microphones and OS's are optimized for human voice and features like automatic gain control aim to reduce any deviations from a normal human voice. Therefore, the higher frequencies, which were not in a normal person's vocal range, may be altered by the operating system or may not be captured accurately by the microphone, since they lay beyond its required standards (3).

For the discrepancies in the peaks, a source for this could be the plot tracing technique, which was manual and not computerized, leaving scope for human error. Another potential source is the tuning of the string. While the instrument was tuned, and the sound was analyzed using the same device to prevent differences in the hardware, the device's scope for inaccuracy in both the hardware and the two applications – the tuning app and the spectrum analyzing app – could have resulted in these errors.

There were two ways of considering the error propagation: the error in theoretical values and the error in the measured values. The uncertainty in a theoretical fundamental frequency (A) of ΔA can be propagated for harmonic frequency f, where $f = kA$, as $\Delta f = k \times \Delta A$.

Here, k is an integer that takes the value of the harmonic. (for example, for the 3rd harmonic $k = 3$).

This way, if the first harmonic had an uncertainty of 1 Hz, the second would be predicted to have an uncertainty of 2 Hz, the third would be predicted to have an uncertainty of 3 Hz, and so on for the theoretical values. In this way, the theoretical model predicts higher uncertainty at higher frequencies.

The theoretical value was based on the idea that the guitar tuner tunes the instrument to 330 Hz. The uncertainty in this value could not be determined. However, if we considered the uncertainty in the fundamental frequency of the first data set, we could get a decent approximation of the margin of error of this 330 Hz fundamental value. The three values for the fundamental frequency were measured as 328 Hz, 331 Hz and 328 Hz. This gave a mean value of 329 Hz with a standard deviation of 1.73 Hz – this was the uncertainty. This uncertainty was used to predict the uncertainty of the 29th harmonic using equation 5. We calculated an uncertainty of 50.17 Hz, which can be taken as the approximate uncertainty in the theoretical frequency value of the 29th harmonic (9570 Hz). Consequently, the theoretically calculated frequency is 9570 ± 25.09 Hz, and the measured frequency is 9608 ± 10.21 Hz. Thus, the mean measured value is very close to what was predicted. The remaining discrepancy might be attributable to nonlinearities of the physical string, but this is outside the scope of the present study.

The uncertainty in the peak width needs consideration as well. The peaks were not exactly sharp, but instead showed a short flat region when zoomed in. The approximate middle of these peaks was taken, and the corresponding frequency was noted, leading to additional error that is not directly assessed in the calculation above.

While the methodology focused on string harmonics, instrument strings are not the only application of higher harmonics. In practice, standing waves occur almost everywhere, from electrical systems to wind blowing through a narrow tunnel. Being able to predict all resonant frequencies, which can potentially harm an electronic device or the structure of a tunnel, is key to successful engineering. As for music, musicians capitalize on higher harmonics for better, more varied tones, which can be utilized to create dynamic songs.

METHODS AND MATERIALS

Node enforcer

To ensure consistency in the excited mode created by enforcing a node, we designed a node-enforcer. The node enforcer uses the concept of dampening strings to enforce points where oscillation is restricted.

The enforcer (see **Figure 3**) was 3D printed in Acrylonitrile butadiene styrene (ABS) plastic. After this, the side with the central cut was covered in paper tape. The cut in the base

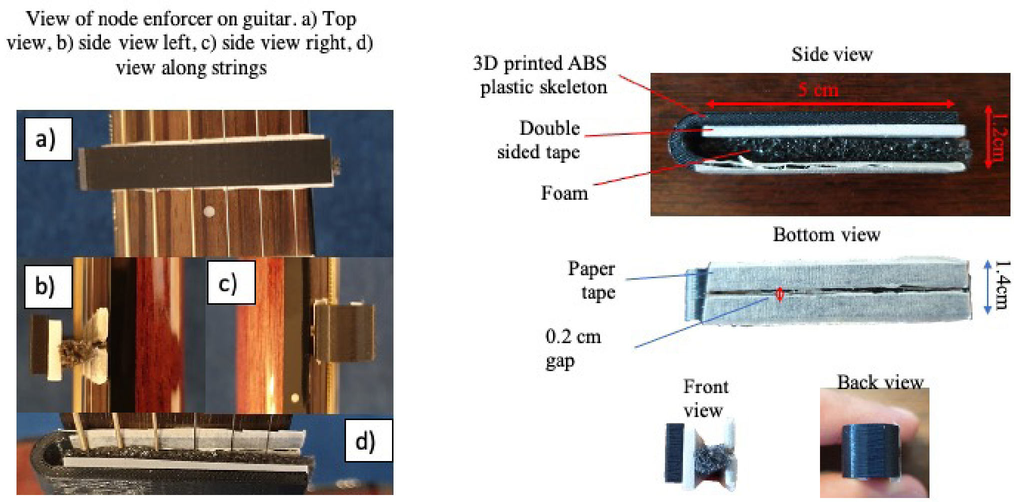

Figure 3. Dimensions of the node enforcer. The node enforcer that was used to enforce the artificial nodes for the study and was designed to fit the guitar used.

allowed it to be clipped onto a fret without any change in the height of the string (and thus maintain constant length, tension, and wave speed). Then, a thin piece of foam was stuck to the other upper end, such that it reached the lower end. The foam acted as a dampener and prevented any oscillation at the point of contact by absorbing the energy at that point, enforcing a node.

Spectrum analyzer

We found the loudest frequencies in the spectrum by ‘pausing’ the sampling of the spectrum in time and then using the inbuilt tracing feature to trace the plot and determine the individual frequencies for each peak manually.

The spectrum analyzer used was the *Advanced Spectrum analyzer PRO by Vuche.* This FFT (Fast Fourier transform) software, with the input samples set at 16384, the averaging factor at 3, and the sampling frequency at 48000 Hz, converted sound into a logarithmic scale graph. The app had “peak hold” and “graph tracing” features, which allowed for both identifying and quantitively obtaining the loudest frequencies - the higher harmonics.

Guitar

The sound was produced using a mahogany body guitar with a 25.5-inch-long scale (scale is the string length), walnut fingerboard, stainless steel frets, walnut bridge, and a crème plastic nut (4). The string used was a bronze string with a gauge size of 0.10 (diameter of 2.542 mm). The manufacturer states that these bronze strings were formed by wrapping an 80% copper and 20% zinc wire around hex shaped, brass plated steel core wire (5). The high E string was played using a Duralin, 1.00 mm thick plectrum. The frequencies were recorded with relative loudness, as loudness is subject to how hard one plucks the string, and this varies from human to human. Moreover, the exact position of the node is subject to error, thus repetitions were performed to minimize this random error.

In summary, the method to record harmonic peaks was as follows: 1) note the background noise, 2) enable peak hold, 3) play a note, 4) pause the spectrum, and 5) use the tracer to find the frequencies of the peaks (loudest) in the spectrum. This was performed thrice for each boundary condition, and the average of these was taken for **Table 2** and the standard deviation of these three values was the uncertainty in the measurement.

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