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Article

An Analysis of the Density and Patterns of the Solutions of Diophantine Equations of the Third Power

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SUMMARY

According to Fermat's last theorem, $x^n + y^n = z^n$ has no solutions if $n > 2$. We modified Fermat's equation into the Diophantine equations $a^3 + b^3 + c^3 = d^3$, $a^3 + b^3 + c^3 = d^2$ and $a^3 + b^3 + c^3 = d^4$ and found their solutions. We analyzed how the density of solutions varied as the numbers got bigger. Ramanujan had devised a formula to find numbers satisfying $a^3 + b^3 + c^3 = d^3$, we compared the density of solutions with those obtained by his formula. We also found perfect cubes, squares or fourth power solutions that could be expressed in different ways as a sum of three cubes. We called them perfect power taxicab numbers. Our hypothesis was that there are many solutions for our equations, and as the inputs become bigger, their density will increase linearly with minor fluctuations. We thought that most perfect power taxicab numbers would have a frequency (number of ways it can be expressed as a sum of three cubes) of two and the maximum frequency would be around 10. We hypothesized that Ramanujan's formula would give around half of the solutions, and the density of solutions will increase as the numbers become large. We concluded that the density distribution of two equations increases as the numbers become bigger. However, the third equation had a stagnant density. Ramanujan's formula found many numbers at the start but was unable to reach a high density. One perfect cube taxicab number had a frequency of 42 whereas the majority had a frequency of 2 or 3.

INTRODUCTION

Since the dawn of civilization, we have been obsessed with finding patterns in the world around us. Our ancestors created mathematics to solve daily problems but were fascinated to find never-ending patterns in numbers. The entire branch of number theory is focused on finding interesting patterns in integers.

Diophantine equations are polynomial equations with more than one variable in which we seek only integral solutions. An exponential Diophantine equation is an equation with exponents on the variables (1). Pythagoras's theorem states that the lengths of the sides of a right-angle triangle satisfy the Diophantine equation $b^2 + p^2 = h^2$. We know that this equation has infinite solutions (2).

Pierre de Fermat was one of the greatest number theorists of the 17th century. On the margin of a copy of the ancient Greek book Arithmetica, he conjectured that the Diophantine equation $x^n + y^n = z^n$ has integer solutions only if n is less than or equal to 2 (3). This statement came to be known as Fermat's last theorem. For example, Fermat's last theorem states that the Diophantine equations $x^3 + y^3 = z^3$ or $x^4 + y^4 = z^4$ do not have any integral solutions. This conjecture turned out to be extremely difficult to prove and this proof eluded mathematicians for centuries. This conjecture was finally proven in 1995 in a 129-page paper by Sir Andrew Wiles (4). Ramanujan, the famous Indian mathematician, contributed to several fields of mathematics including number theory, continued fractions, mathematical analysis and infinite series (5,6). His works continue to be studied and analyzed by mathematicians even a century after he died. Many mathematicians including Ramanujan devised formulas to find numbers satisfying $a^3 + b^3 + c^3 = d^3$ (7,8). There is no known formula that can find all the solutions of the above equation (9).

In the famous exchange when G. H. Hardy came to visit Ramanujan in the hospital, Hardy said that the cab he came to meet him in was numbered 1729. Hardy remarked it was an extremely dull number (10). Ramanujan however, answered that it was a very interesting number. Ramanujan said that this was the smallest integer which could be expressed as the sum of two cubes in two different ways:

$$1729 = 1^3 + 12^3$$
$$1729 = 9^3 + 10^3$$

Numbers of this kind came to be known as taxicab numbers or Hardy-Ramanujan numbers (11).

In this paper we have varied the equation in Fermat's last theorem into Diophantine equations of the form $a^3 + b^3 + c^3 = d^3$, $a^3 + b^3 + c^3 = d^2$ and $a^3 + b^3 + c^3 = d^4$ and found integral solutions for them keeping a, b and c to less than 10,000. Using a computer program that we wrote, we found many sets of integers satisfying the above three equations by brute force – putting in each number one by one and only keeping the sets that satisfy the equations. After we found all the solutions under 10,000, we analyzed the density of solutions and how it varied as the numbers got bigger.

We wanted to find out how many solutions Ramanujan's formula could find, and we wanted to compare the density of solutions got by his formula to our own calculated density.

To accomplish this, we made a computer program to find the solutions of $a^3 + b^3 + c^3 = d^3$ using his formula and compared the density of solutions of those obtained by his formula with those we found.

In this paper, we generalized Hardy-Ramanujan's taxicab numbers to a perfect power able to express itself as a sum of three cubes in different ways. For example, the number 1,000,000 is a perfect cube and can be expressed in two ways as a sum of three cubes:

$$1{,}000{,}000 = 100^3 = 85^3 + 70^3 + 35^3$$
$$1{,}000{,}000 = 100^3 = 88^3 + 68^3 + 16^3$$

So, we say that 1,000,000 is a perfect cube taxicab number and has a frequency of 2.

We found perfect square, perfect cube and perfect fourth power taxicab numbers keeping *a*, *b* and *c* to less than 10,000.

Our perfect power taxicab numbers are different from Ramanujan's taxicab numbers in the way that our numbers are the sum of three and not two cubes and the result itself is a perfect cube, perfect square or perfect fourth power rather than any integer as in Ramanujan's case. We found many perfect cubes, square and fourth power taxicab numbers according to the equation. We analyzed and compared the frequency of perfect power taxicab numbers according to the number of times they could be expressed in different ways.

Our hypothesis was that there are many solutions for our equations, and as the inputs become bigger, their density will increase linearly with minor fluctuations. We thought that most perfect power taxicab numbers would have a frequency of two and the maximum frequency would be around ten. We hypothesized that Ramanujan's formula would give around half of the solutions, and the density of solutions will increase as the numbers become large.

RESULTS

We analyzed the solutions of the three equations $a^3 + b^3 + c^3 = d^3$, $a^3 + b^3 + c^3 = d^2$ and $a^3 + b^3 + c^3 = d^4$ and compared their densities.

From our data we found interesting patterns in all the three equations. The number of solutions of the equation $a^3 + b^3 + c^3 = d^3$ in which all *a*, *b* and c are less than or equal to 10,000 is 65,085. Throughout our analysis, we assumed that $a \geq b$ and c. Therefore, we used a (the largest of the three inputs) as the basis of our density analysis. We found that the number of solutions of $a^3 + b^3 + c^3 = d^2$ for *a*, *b* and c less than or equal to 10,000 was 266,517 and that of the equation $a^3 + b^3 + c^3 = d^4$ for *a*, *b*, and *c* less than or equal to 10,000 was just 987.

Density of solutions for $a^3 + b^3 + c^3 = d^3$ (Cubic equation)

The total number of solutions for *a*, *b*, and c below or equal to 10,000 were 65,085. The first solution was

$$5^3 + 4^3 + 3^3 = 6^3 = 216,$$

and the last solution in the set of numbers that we considered was

$$9999^3 + 9696^3 + 9494^3 = 14039^3 = 2{,}766{,}995{,}941{,}319.$$

We used our data to analyze how the density of solutions to this equation varies as the input numbers (*a*, *b*, and *c*) become bigger. To visualize the density of solutions, we looked at the solutions in two different ways. We considered the number of solutions for every 100 multiples of the largest of *a*, *b*, and *c* in our equation. In our analysis we always took $a \geq b$ and c. For example, between $a = 0$ and $a = 100$, there were 98 solutions. Between $a = 100$ to $a = 200$, there were 183 solutions, and so on (**Figure 1a**). We also analyzed how the number of solutions per number of combinations of *a*, *b* and *c* varies per 100 multiples of *a* (**Figure 1b**).

We could see that the number of solutions per hundred values of a (density of solutions) rose quite steeply at first and then the rise became more gradual (**Figure 1a**). As we approached $a = 10{,}000$, the density reached more than eight hundred solutions for every 100 values of *a*. There were some minor fluctuations, but the overall trend was that of a rapid increase in the beginning and a slower increase later. The maximum number of solutions was found between $a = 9000$ and $a = 9100$. This range has 902 solutions. The minimum number of solutions was between $a = 0$ and $a = 100$: 98 solutions (**Figure 1a**).

As *a*, *b* and *c* increased the total number of solutions per combinations of *a*, *b* and *c* decreased rapidly (**Figure 1b**). In a sense total number of solutions per combinations

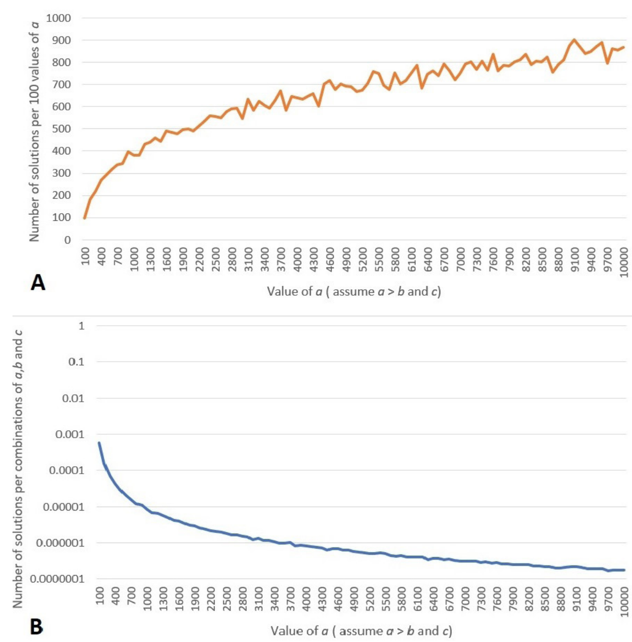

Figure 1: Solution density for the cubic Diophantine equation. A) Number of solutions per hundred numbers of *a* for the equation $a^3 + b^3 + c^3 = d^3$. The maximum number of solutions is found between $a = 9000$ and $a = 9100$. This range has 902 solutions. The minimum number of solutions is between $a = 0$ and $a = 100$: 98 solutions. B) The number of solutions per number of combinations of *a*, *b* and *c* vs values of *a* with bin size of 100 ($a \geq b$ and *c*).

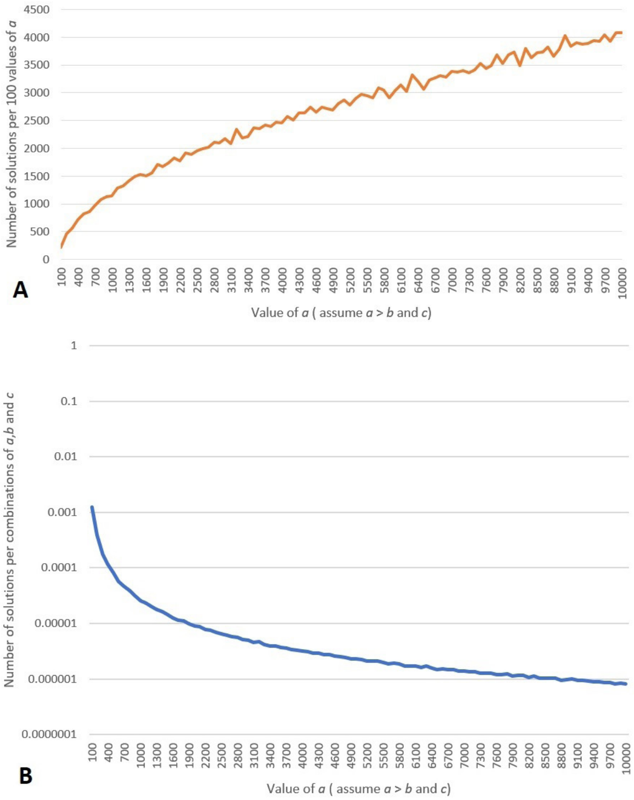

Figure 2: Solution density for the square Diophantine equation. A) Number of solutions per hundred numbers of *a* for the equation $a^3 + b^3 + c^3 = d^2$. The minimum density in the range of numbers that we considered was between *a* = 0 to *a* = 100, which was 214. The maximum density was between *a* = 9,800 to *a* = 9,900, which was 4,078. B) The number of solutions per number of combinations of *a*, *b* and *c* vs values of *a* with bin size of 100.

of the inputs (**Figure 1b**) is a better measure of the density compared to looking at just the largest input (**Figure 1a**). The latter only focuses on the largest variable *a*. It does not take into account the fact that as a increases, the number of *b* and *c* values that can be tried also increase. However, **Figure 1b** normalizes the density by finding the total number of solutions in any given range of a divided by number of combinations of *a*, *b* and *c* that will be tried in that range of *a*.

Density of solutions for $a^3 + b^3 + c^3 = d^2$ (Square or quadratic equation)

The total number of solutions below ten thousand values of a was 266,517. The first solution was

$$3^3 + 2^3 + 1^3 = 6^2 = 36,$$

and the last solution in our data was

$$9999^3 + 9945^3 + 6390^3 = 1498068^2 = 2{,}244{,}207{,}732{,}624.$$

We used our data to analyze how the density of solutions to this equation varied as the input numbers (*a*, *b* and *c*) become bigger. We considered the number of solutions for every 100 multiples of the largest of *a*, *b* and *c* in our equation (**Figure 2a**). In our analysis we always took *a* ≥ *b* and *c*. The minimum density in the range of numbers that we considered was between *a* = 0 to *a* = 100, which was 214. The maximum density was between *a* = 9,800 to *a* = 9,900, which was 4,078. From the plot we concluded that as the value of a increased, the density of solutions also steadily increased.

We also analyzed how the number of solutions per combinations of *a*, *b* and *c* varied per 100 multiples of *a* (**Figure 2b**). The number of solutions per combinations of *a*, *b* and *c* dropped quite rapidly at first and then flattened out.

Density of solutions for $a^3 + b^3 + c^3 = d^4$ (4th power equation or quartic equation)

The total number of solutions of this equation within the first 10,000 values of *a*, *b* and *c* is only 987. The first solution was

$$3^3 + 3^3 + 3^3 = 3^4 = 81$$

and the last solution within the numbers that we considered was

$$9986^3 + 4544^3 + 3006^3 = 1028^4 = 1{,}116{,}792{,}422{,}656$$

The total number of solutions decreased from the quadratic to cubic to quartic (266517 vs 65085 to 987), and that was expected because the total count of numbers whose square is less than 3×10^{12}(*a*, 10^{12} values of *b* and 10^{12} values of *c*) is much more than the total number of cubes in the range which, in turn, is much greater than the total number of powers of four in the range.

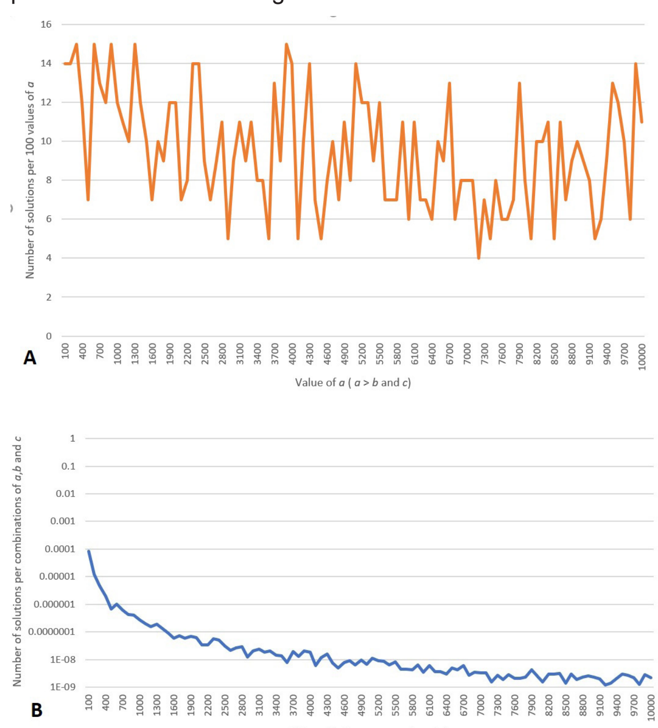

Figure 3: Solution density for the quartic Diophantine equation. A) Number of solutions per hundred numbers of *a* for the equation $a^3 + b^3 + c^3 = d^4$. The maximum solution density is 15, which is achieved by five ranges of *a*. The minimum frequency density was between *a* = 7100 and *a* = 7200, which is four. This equation's solution density shows a lot of local fluctuations but has a constant "flat" trend. B). The number of solutions per number of combinations of *a*, *b* and *c* vs values of *a* with bin size of 100.

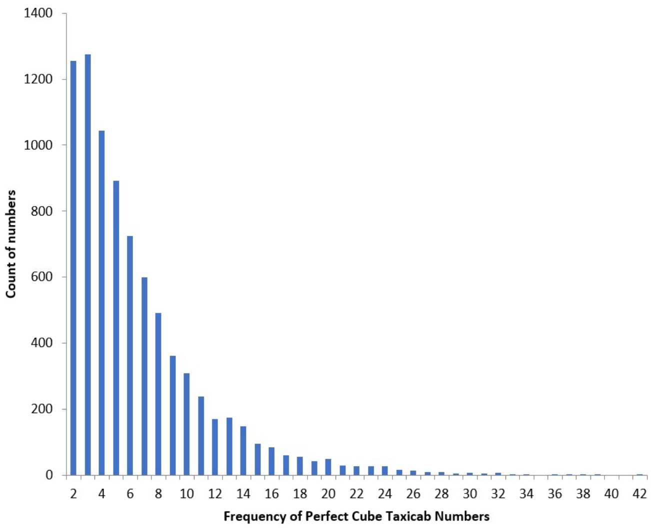

Figure 4: Histogram which compares the count of numbers with different frequencies for the equation $a^3 + b^3 + c^3 = d^3$. The count of numbers with frequency of 3 is greater than count of numbers with frequency of 2. The maximum frequency is 42 for perfect cube taxicab numbers

We analyzed the solution density of this equation for the first 10,000 values of *a* (assume $a \geq b$ and *c*) (**Figure 3a**). We also considered how the number of solutions per number of combinations of *a*, *b* and *c* varied per 100 multiples of *a* (**Figure 3b**).

We inferred (**Figure 3a**) that this equation's solution density was unlike that of the previous two equations (the square and the cubic). Here the maximum solution density was 15, which was achieved by five ranges of a. The minimum density was between $a = 7100$ and $a = 7200$, which was four. This equation's solution density showed a lot of local fluctuations but had a constant "flat" trend. The first range $a = 0$ to $a = 100$ had the highest density of 15, and this was repeated 4 times. This curve was constrained within the range 4-15 with large fluctuations. In the square and cubic results (Figures 1a and 2a), the value of the density went up to thousands of numbers while the highest density for the quartic equation only went up to 15.

When we analyzed how the number of solutions per number of combinations of *a*, *b* and *c* varied per 100 multiples of *a* we saw that this shows a different character than the corresponding solutions of the square and cubic equation (**Figure 3b**). The fall was faster and the flattening happened later compared to the square and the cubic equation. The density also showed significant local variations.

A very effective way to compare the three equations was to fit some simple polynomial curves on the data and compare these curves. We found that the following three curves fit into the data of number of solutions per 100 values of *a* (Figures 1a, 2a and 3a):

Cubic equation (**Figure 1a**):

$$-6.312\times10^{-6}\, x^2 + 0.1199x + 258.9$$

Square equation (**Figure 2a**):

$$-2.145\times10^{-5}\, x^2 + 0.5471x + 627.6$$

4th power equation (**Figure 3a**):

$$1.132\times10^{-7}\, x^2 - 0.001511x + 13.35$$

When we compare these three polynomials by plotting them and by algebraic analysis, we saw that the fourth power polynomial was almost flat. The third power curve had a more gradual slope than the 2nd power curve and did not rise as high.

The following curves fit the data of the number of solutions per combinations of *a*, *b* and *c* plotted against values of a with bin size of 100 (Figures 1b, 2b and 3b)

Cubic equation (**Figure 1b**):

$$3.365\times10^{-8}\, x^2 - 0.0005524\, x - 4.393$$

Square equation (**Figure 2b**):

$$3.084\times10^{-8}\, x^2 - 0.0005014\, x - 3.965$$

4th power equation (**Figure 3b**):

$$4.69\times10^{-8}\, x^2 - 0.0007486\, x - 5.683$$

When we compared these polynomials we saw that the fourth power curve fell much more rapidly compared to the other curves and reached a much lower point than the other two curves. The flattening of this curve happened later compared to the cubic equation and square equation curves. The cubic curve and the square equation curve had similar slopes and shape but the cubic curve was lower than the square equation curve but still much higher than the 4th power curve.

Perfect Power Taxicab numbers

We defined perfect power taxicab numbers as perfect cubes, squares or fourth powers satisfying the Diophantine equations outlined above, which can be expressed as a sum of three cubes in different ways. Surprisingly we found many perfect power taxicab numbers for all the three Diophantine equations that we considered.

We found perfect cube taxicab numbers i.e. multiple *a*, *b* and *c* for the same *d* satisfying $a^3 + b^3 + c^3 = d^3$ and compared the count of numbers for each frequency (**Figure 4**). We

970299000000 $= 9900^3$		
$= 7590^3 + 6490^3 + 6380^3$	$= 7625^3 + 6775^3 + 6000^3$	$= 7704^3 + 6796^3 + 5840^3$
$= 7911^3 + 7713^3 + 2538^3$	$= 8019^3 + 6822^3 + 5157^3$	$= 8040^3 + 7512^3 + 2988^3$
$= 8145^3 + 7240^3 + 3695^3$	$= 8205^3 + 6030^3 + 5835^3$	$= 8208^3 + 6156^3 + 5688^3$
$= 8240^3 + 6540^3 + 5080^3$	$= 8250^3 + 6600^3 + 4950^3$	$= 8310^3 + 6810^3 + 4320^3$
$= 8355^3 + 6885^3 + 3930^3$	$= 8415^3 + 6930^3 + 3465^3$	$= 8485^3 + 6939^3 + 2936^3$
$= 8487^3 + 6504^3 + 4377^3$	$= 8500^3 + 6750^3 + 3650^3$	$= 8586^3 + 6936^3 + 1542^3$
$= 8614^3 + 6848^3 + 2154^3$	$= 8640^3 + 6750^3 + 2610^3$	$= 8646^3 + 6858^3 + 1128^3$
$= 8712^3 + 5676^3 + 5016^3$	$= 8712^3 + 6732^3 + 1584^3$	$= 8766^3 + 6108^3 + 4098^3$
$= 8800^3 + 6600^3 + 1100^3$	$= 8982^3 + 6186^3 + 2076^3$	$= 9036^3 + 6108^3 + 1668^3$
$= 9108^3 + 4818^3 + 4686^3$	$= 9120^3 + 5170^3 + 4190^3$	$= 9225^3 + 5175^3 + 3600^3$
$= 9225^3 + 5700^3 + 375^3$	$= 9231^3 + 4593^3 + 4428^3$	$= 9384^3 + 4491^3 + 3765^3$
$= 9416^3 + 4785^3 + 2959^3$	$= 9475^3 + 4925^3 + 600^3$	$= 9570^3 + 4180^3 + 2750^3$
$= 9609^3 + 4054^3 + 2543^3$	$= 9636^3 + 4180^3 + 1364^3$	$= 9675^3 + 3725^3 + 2350^3$
$= 9719^3 + 3313^3 + 2514^3$	$= 9850^3 + 2300^3 + 1350^3$	$= 9890^3 + 1360^3 + 750^3$

Figure 5: 42 ways in which a perfect cube could be expressed as a sum of three different cubes.

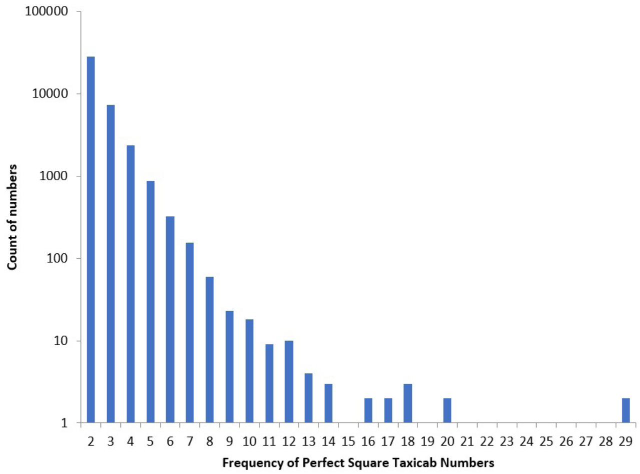

Figure 6: Histogram which compares the count of numbers with different frequencies for the equation $a^3 + b^3 + c^3 = d^2$. The maximum frequency is limited to 29 for perfect square taxicab numbers.

define the frequency of a taxicab number for the equations we considered as the number of different ways a perfect cube, square or fourth power taxicab number is expressed as a sum of three cubes. For example:

$$1{,}000{,}000 = 100^3 = 85^3 + 70^3 + 35^3$$
$$1{,}000{,}000 = 100^3 = 88^3 + 68^3 + 16^3$$

Therefore, we say that 1,000,000 is a perfect cube taxicab number and has a frequency of 2. There are 1256 taxicab numbers which have a frequency of 2 for the equation $a^3 + b^3 + c^3 = d^3$. Thus, we say that the count of numbers that have the frequency of 2 is 1256.

Our calculations allowed us to find all perfect cube taxicab numbers up to $10{,}000^3$ ($d^3 \leq 10{,}000^3$). It may seem that our analysis should have found all perfect cube taxicab numbers up to $3 \times (10{,}000)^3$ since a ranges from 1 to 10,000 and b and c range from 0 to a, but that is not the case. There will be taxicab numbers less than $3 \times (10{,}000)^3$ but greater than $10{,}000^3$ which have $a > 10{,}000$ but b and $c < 10{,}000$. On the other hand, we can prove that all taxicab numbers up to $10{,}000^3$ have been covered in our analysis. This can be seen from the fact that the maximum value of a is 10,000 and the minimum value of b and c are 0. In this limiting case the equation becomes $10{,}000^3 + 0^3 + 0^3 = 10{,}000^3$. Thus, our analysis of a, b and c numbers where each of them is less than or equal to 10,000 does cover all perfect cube taxicab numbers up to $10{,}000^3$ but does not cover all perfect cube taxicab numbers $> 10{,}000^3$ as some of them could have $a > 10{,}000$ but b and $c < 10{,}000$. Therefore, although we had found some frequencies of taxicab numbers $d^3 > 10^{12}$ in our data, we did not consider it in our analysis as this data is incomplete for the above-mentioned reasons.

In total, there were 8276 different perfect cube taxicab numbers up to a = 10,000. Most of these numbers had a frequency of 2 and 3 but we could find many numbers whose frequency in this data went beyond twenty (**Figure 4**). The maximum frequency for a perfect cube taxicab number was 42. So, there were 42 ways in which a perfect cube could be expressed as a sum of three different cubes (**Figure 5**). We had expected that the count of numbers with frequency of 2 will dominate but we were surprised to see that the count of numbers with frequency of 3 was greater than the count of numbers with frequency of 2.

We then compared the count of numbers with different frequencies for the equation $a^3 + b^3 + c^3 = d^2$ (**Figure 6**). These are perfect square taxicab numbers i.e. multiple a, b and c for same d satisfying the equation. As described for the cubic Diophantine equation above, our data consists of all taxicab numbers such that $d^2 \leq 10{,}000^3$ (i.e. $d < 10^6$).

We inferred that although the number of perfect square taxicab numbers (39,153) is far greater than that of the perfect cube taxicab numbers (8276), the frequency is limited to a certain range (**Figure 4**, **Figure 6**). The maximum frequency is limited to 29 for perfect square taxicab numbers whereas the perfect cube taxicab numbers peaked at a frequency of 42. Another difference from the cubic case was that for the square equation, the count of numbers with frequency of 2 far exceeds the count with the frequency of 3.

We analyzed the frequency of perfect fourth power taxicab numbers, i.e. multiple a, b and c for same d satisfying $a^3 + b^3 + c^3 = d^4$ (**Figure 7**). We compared the count of numbers with different frequencies. As described in the cubic Diophantine equation above, we found all taxicab numbers

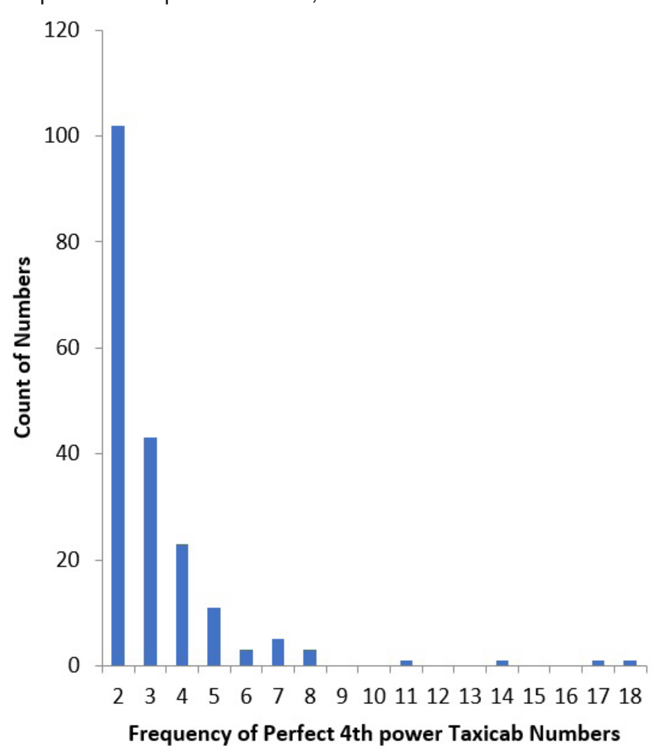

Figure 7: Histogram which compares the count of numbers with different frequencies for the equation $a^3 + b^3 + c^3 = d^4$. The maximum frequency of perfect fourth power taxicab numbers in the first 10,000 numbers was 18.

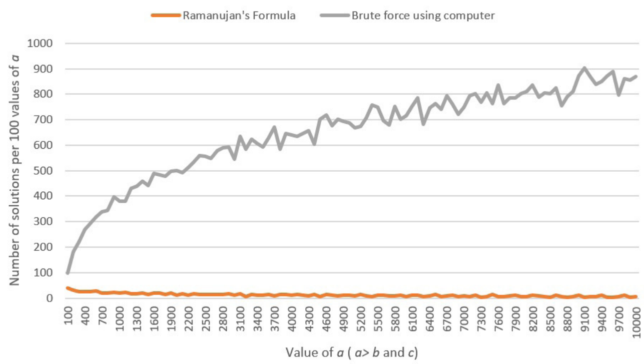

Figure 8: Comparison of the density of solutions of $a^3 + b^3 + c^3 = d^3$ got by Ramanujan's formula and our computer program (brute force method). Ramanujan's formula could only find a fraction of the solutions that we found by brute force, and as the numbers grew larger, the gap became bigger.

such that $d^4 = 10{,}000^3$ (i.e. $d = 10^3$).

We could see that there are very few fourth power taxicab numbers in the first ten thousand numbers of *a* (assuming $a \geq b, c$) (**Figure 7**). However, we saw that out of 987 solutions of this equation in the range of numbers we considered, there were 194 different perfect fourth power taxicab numbers. Therefore, the ratio of taxicab numbers to total solutions in the 4th power equation is much greater than the ratio in the square equation and the cubic equation. The maximum frequency of perfect fourth power taxicab numbers in the first 10,000 numbers was 18.

Ramanujan's formula for Diophantine equations of the third power

Ramanujan proposed a formula to find solutions of $a^3 + b^3 + c^3 = d^3$ (7,8) which is as follows:

$$(3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 = (6x^2 - 4xy + 4y^2)^3$$

Here $(3x^2 + 5xy - 5y^2)$, $((4x^2 - 4xy + 6y^2)$ and $(5x^2 - 5xy - 3y^2)$ are our *a*, *b* and *c*, where the largest number for a specific *x*, *y* is *a*, second largest number is *b* and the smallest is *c*. To generate the solutions obtained by the above equation, we inserted values of *x* and *y* such that *a* would range from 0 to 10,000. Here both *x* and *y* can be negative but we selected only those solutions for which *a*, *b*, *c* and *d* were positive. We plotted the density of solutions of $a^3 + b^3 + c^3 = d^3$ generated by Ramanujan's formula and compared this data with our own data for a less than or equal to 10,000 (**Figure 8**). For this analysis we considered the number of solutions per hundred numbers of *a* and compared the density obtained from Ramanujan's formula to the density obtained by our program. A complete density analysis would also require us to divide the number of solutions in a given interval by the number of combinations of *a*, *b* and *c*. However, our main aim in this analysis was to compare Ramanujan's formula with the number of solutions found by brute force and that comparison would be equally relevant if we were just to consider the number of solutions per 100 values of *a*.

When *a* was small, the gap between the density of solutions found by Ramanujan's formula and our program was about 3 times: 38 vs 98 in the first 100. However, the density of solutions in the two cases quickly diverged. With the increase in *a*, the density of solutions of Ramanujan's formula was unable to generate enough numbers and the density became flat and the gap between the two densities kept increasing. The total number of solutions found by Ramanujan's formula in the first 10,000 values of *a* were 1195. This was very small compared the numbers found by brute force: 65,085. Therefore, we concluded that while Ramanujan's formula does generate numbers satisfying the equation, it fails to generate enough density or count compared to the method of brute force.

DISCUSSION

In this paper we found solutions of following three Diophantine equations of the third power by brute force for the first ten thousand numbers of a.

$a^3 + b^3 + c^3 = d^2$ (quadratic equation or square equation)
$a^3 + b^3 + c^3 = d^3$ (cubic equation)
$a^3 + b^3 + c^3 = d^4$ (quartic equation or 4th power equation)

Here *a* is the largest number on the left-hand side of all the three Diophantine equations. We compiled all the solutions for the three equations into graphs showing density of solutions per hundred numbers of a and compared them. We found that while the quadratic equation and cubic equation showed a similar trend in density as the numbers became large (they rose rapidly at first and later the rise slowed down), the quartic equation's density was constant and constrained. The quartic equation's density at any point was much less than the density of the quadratic and cubic equations at that point.

We also compared the total number of solutions per combinations of *a*, *b* and *c* as *a* increased from 1 to 10,000. This graph fell rapidly at first in all the three cases and then stabilized. The fall was much larger for the fourth power equation compared to the cubic and quadratic equation.

We also found that the quadratic equation produced the greatest number of solutions followed by the cubic equation, whereas the quartic equation produced less than thousand solutions in our data.

We found numbers which we called perfect power taxicab numbers that are perfect squares, cubes or 4th powers, according to the equation, that could be expressed as a sum of three cubes in different ways. We compiled the perfect power taxicab numbers into histograms to find the count of numbers corresponding to various frequencies. We found that the number of perfect square taxicab numbers was much more than the number of perfect cube taxicab numbers. The number of perfect 4th power taxicab numbers was the least. Surprisingly, one perfect cube taxicab number

(970,299,000,000) could be expressed as the sum of three cubes in 42 different ways. In the cubic equation we were surprised to find that the count of numbers with frequency of 3 was more than the count of numbers with frequency of 2.

Finally, we generated solutions of the cubic equation given by Ramanujan's formula and compared its density with ours. We found that Ramanujan's formula could not match the density or count of solutions found by us using brute force.

When we compared our results with our hypothesis, we found that our hypothesis was correct in that the number of solutions rose with an increase in a for the square and the cubic equation. For the 4th power equation, however, the density fluctuated rapidly but stayed stable as a increased. We did not expect to see any taxicab number that could have a frequency of more than 20. However, in the cubic case we found a number (970,299,000,000) with the frequency of 42. Ramanujan's formula to find solutions of the cubic equation could only find a small fraction of the solutions that we found by brute force, and as the numbers grew larger, the gap became bigger. This trend was not what we expected in our initial hypothesis.

In this paper, we showed the trends of three Diophantine equations of the third power of the three input variables. We have merely explored a small area of number theory of Diophantine equations. An idea for a future research topic would be to analyze and compare the density of $a^3 + b^3 + c^3 + d^3 = e^3$ with the equations that we have studied. However, as the number of variables in the Diophantine equation increases, it takes up a lot of computation time to find all the solutions. We tried our Python programs with the four variable Diophantine equation. We ran the program for this equation for three days but the program did not even reach a = 1000. The reason for this is that the number of loops increases with the addition of another variable. So, if one Diophantine equation takes 1 hour to finish finding the solutions up to a = 10,000, the time taken to finish the same Diophantine equation with an added variable will increase by a factor of ten thousand (i.e. take 10,000 hours), and so on. Therefore, this is a good idea for a future research paper in which this equation can be tackled by better algorithms or by using powerful processors and distributed programming. We could also find solutions for all the above Diophantine equations till a equals one million to see how the density graphs change after a = 10,000. However, it would require faster computers and more efficient algorithms.

MATERIALS AND METHODS

In our paper we utilized the power of computer programming to create and evaluate our data. We wrote Python programs (Python version 3.7) for finding all our numbers for each Diophantine equation using brute force. By brute force we mean that we tried every combination of numbers to find the solutions to the equations. We redirected the output into a text file arranged as columns of a, b, c and d. We designed a separate Python program to parse that data into density per hundred numbers of a. To find the density of numbers we arranged all solutions in increasing order of a. Then another program found the difference of the line numbers of every consecutive multiple of 100 as. In order to find curves that fit our data for easy comparison we used polyfit and poly1d functions that are available in the numpy scientific package of Python.

To find perfect power taxicab numbers, we created Python programs that would take every d and tried to find if the same d could be expressed in different ways. It did this by parsing our file and matching the d values.

To generate the solutions obtained by Ramanujan's equation we needed to find x and y such that the equation $(3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3 = (6x^2 - 4xy + 4y^2)^3$ could yield solutions within the range that we had selected: a ranging from 0 to 10,000 and $a \geq b$ and c. To accomplish this, we considered the three contributing terms: $3x^2 + 5xy - 5y^2$, $4x^2 - 4xy + 6y^2$ and $5x^2 - 5xy - 3y^2$. Each of these had to be positive integers and less than 10,000. So, we plotted the following inequations:

$$3x^2 + 5xy - 5y^2 > 0$$
$$3x^2 + 5xy - 5y^2 < 10000$$
$$4x^2 - 4xy + 6y^2 > 0$$
$$4x^2 - 4xy + 6y^2 < 10000$$
$$5x^2 - 5xy - 3y^2 > 0$$
$$5x^2 - 5xy - 3y^2 < 10000$$

Out of the above only the curves corresponding to $4x^2 - 4xy + 6y^2$ were bound and closed. From this we could determine the values of x needed to calculate the solutions to the cubic equation in the range of values that we had selected. We found that the x should range from -55 to +55. The plot of the other equations showed us that y should always be less than x. We implemented a Python program which we looped x and y from -55 to 55 and selected only those solutions for which a, b, c and d were positive and the largest of the three inputs is less than or equal to 10,000; the rest of the numbers were discarded.

We arranged all our data output by Python programs separated by '|', so it was easier to input this data into Microsoft Excel (version 1902) to create graphs which we could then analyze. Then, we arranged this data per hundred numbers of a and compared its density to that achieved by our data. We also used commands like 'grep' for finding a number in our data, 'sort' for sorting data in an order, 'wc' for counting lines, 'sed' for deleting or adding characters at every line. These commands were available to us as we operated on a Linux machine.

Since our programs were complex and computationally intensive, we purchased an instance on Amazon Web Services for computational time. We needed this to speed up the execution of our programs. While searching for numbers, we were careful about any defects in our data. Initially, there were several problems in our data which we fixed through rigorous searching and debugging. We always kept a backup of old data by making new files at every stage.

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