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Article

Modeling the moving sofas in circular hallways using geometric methods

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SUMMARY

The moving sofa problem is an unresolved geometry problem introduced by Leo Moser in 1966. The problem seeks the largest planar and rigid shape that can move around a right-angled hallway of unit width. While the problem may seem like purely mathematical curiosity, it has many potential applications. Particularly, insights from the problem can be used to design trolleys that can move through tight corners or to navigate large objects in limited spaces. The more generalized piano mover's problem also has practical applications in motion planning, such as robot navigation, robotic surgery, automation, driverless cars, and computer games. Though significant progress has been made in solving the original moving sofa problem, there has not been much study on more generalized cases. Here, we investigated the largest rigid shape that can be moved through a circular hallway of unit width with an arbitrary turn angle. We generated three hypotheses: first, the maximal area of moving sofa should depend on both the radius and the angle of the circular hallway; second, the maximal area should increase monotonically with the radius but decrease monotonically with the angle; and third, the result should resemble that of the original problem with turn angle θ = 90° and inner radius r_i = 0. We found the results of our geometric models align well with the three hypotheses in both cases of inner radius r_i = 0 and r_i > 0. Our findings may provide new perspectives to applications of the moving sofa problem.

INTRODUCTION

In 1966, Leo Moser formally stated the moving sofa problem, which asks for the largest area of a planar and rigid shape that can be moved around a 90° corner of a long hallway with a width of 1 unit (1). Despite the problem being simple to understand, it remains unsolved to this day (2). Besides its real-world applications, solving the problem would offer benefits such as advancing mathematical tools on geometric reasoning and optimization methods.

A trivial example would be a unit square with area 1, which can move around the 90° hallway corner of unit width by pushing it to the end of the first portion of the L-shaped unit-width hallway and then by pulling it out through the second portion of the unit-width hallway **(Figure 1)**. Another trivial example is a semicircle of unit radius with a much larger area of $\pi/2 \approx 1.5708$, which will first have to be pushed it to the end of the first portion of the hallway, then rotated 90° at the hallway corner, and then pulled out in the second portion of the hallway (**Figure 1**). However, better sofas can be constructed that move through with both translations and rotations occurring simultaneously. In 1968, John Hammersley presented the first non-trivial results for the lower bound (LB) sofa area to be $\pi/2 + 2/\pi \approx 2.2074$ and the upper bound (UB) sofa area to be $2\sqrt{2} \approx 2.8284$, respectively (3). The LB and UB, by definition, provide the lowest and highest possible limits to the maximal area of a moving sofa, respectively. While the exact value of the maximal area of the moving sofa is difficult to find and even harder to prove, improvements can be still made by providing a higher value for the LB or a lower value for the UB. Hammersley's LB sofa was composed of two quarter unit circles separated by a $4/\pi \times 1$ rectangle with a half circle of diameter $4/\pi$ cut out. Putting the shapes together this way increased the LB of sofa area significantly, for example, by 10.37% compared to the semicircle. Later in 1992, Joseph Gerver increased the LB to about 2.2195 by modeling the LB sofa area with a geometric envelope with 18 distinct curve segments (4). Gerver was able to add more area at the outer borders after he carved out some areas from the two inner conners of Hammersley's sofa. Since then, no larger LB sofa has been reported in literature, which may indicate that Gerver's sofa is either optimal or very close to it (5). In 2018, however, Kallus and Romik improved the UB sofa area to 2.37 by a computer-assisted proof (6). Based on Gerver's ideas, Romik formulated a moving sofa model in 2016 for the ambidextrous variant of the problem, in which the problem is to find the rigid shape can move around unit-width hallways with 90° turns both to the left and to the right (7-9).

In this study, we considered a variant of the moving sofa problem that uses circular hallways rather than the L-shaped hallway. We first hypothesized that the largest planar areas that can be maneuvered through such unit-width circular hallways must exist when the hallways span out a circular sector with a finite turn angle *θ*. Using the definition, the turn angle has a range of 0° < *θ* ≤ 180° with *θ* = 180° representing a U-turn of the circular hallway. We assumed that the maximal moving sofa areas should depend on both the radii of the circular hallways and their turn angles. Secondly, we hypothesized that the maximal moving sofa areas should increase monotonically with the radii of the circular hallways but decrease monotonically with their turn angles. At the low end of the turn angles when *θ* approaches 0°, the moving sofa areas should increase quickly from some finite values to infinity; but at the high end when *θ* approaches 180°, the decreasing rate should be much slower to some other finite values at 180°. When *θ* = 90° and the hallway inner radius r_i = 0 (i.e., the case of sharp inner corner), our problem statement is quite like the original moving sofa problem (1). Thus, our third hypothesis is that the sofa areas when *θ* = 90° and r_i

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= 0 should be very close to those of the original problem if not the same. Our results align well with all the above three hypotheses. Particularly in the case of the 90° sharp inner corner, our data resemble the original moving sofa problem. Our main contributions include the new geometric models and calculation methods that can be used to approximate the maximal areas for the more generalized moving sofa problems as well as providing the LBs and UBs.

RESULTS

In this study, the primary goals were to find out the maximal areas of moving sofas through geometric models and to determine the LBs and UBs for the more generalized cases.

Sharp Inner Corner Problems

We first investigated the maximal moving sofas that can go through a circular hallway of inner radius r_i = 0 and outer radius r_o = 1 that spans across an arbitrary turn angle θ (0° < θ ≤ 180°) (**Figure 2A**). In such case, the unit-width straight hallway merely rotates around to make a fan shape then turns immediately into another unit-width straight hallway. For the special example when θ = 90°, the hallway is the same as the original moving sofa problem except for a circular outer corner (**Figure 2B**). In another special example when θ = 180°, the unit-width hallway makes a sharp U-turn (**Figure 2C**).

We sought to construct geometric models for the special case where the inner radius of the circular hallway is zero. The model construction was as follows: first, we set the origin point *O* of Cartesian coordinate system locate at the center of rotation fan shape, which is also the sharp corner of the two inner straight hallway walls (**Figure 2A, 3A**). Since the moving sofas must fit in both the two straight portions and the fan portion of the hallway, we considered the intersections of the hallway with infinite bands of unit width for the LB and UB sofas, respectively, which are perpendicular to the symmetric line of the entire hallway. The maximal LB and UB sofas must be contained entirely in the intersections of the hallway with

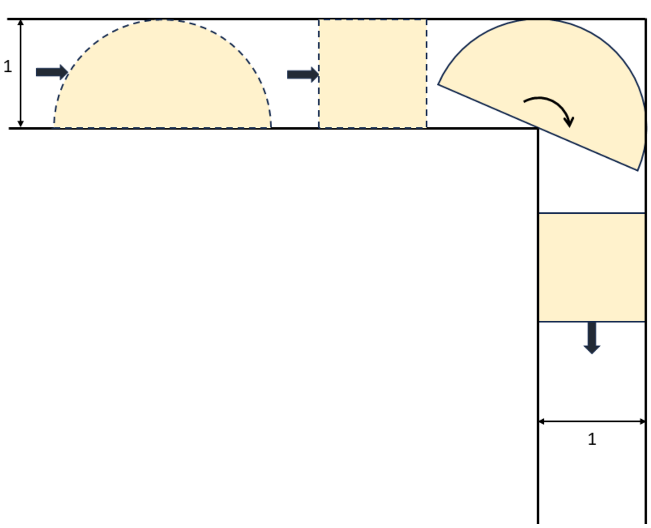

Figure 1: The original moving sofa problem: A unit square and a semicircle of radius 1, with areas 1 and π/2 ≈ 1.5708, respectively, moving through a 90° "L"-shaped hallway of unit width.

the unit bands of LB and UB respectively, when they are moved halfway through.

For the LB determination, when the LB sofa is halfway, the point *O* must be inside the unit band of LB (the blue dashed lines in **Figure 3**). Otherwise, the LB sofa is either not maximal or cannot move through the hallway. In other words, the point *O* is located inside the two intersect point *P* and *Q* of the unit band of LB with the symmetric line of the hallway (**Figure 3**). With the above arguments, the LB sofa area was maximized based on two independent parameters *c* and *d*, where 0 ≤ *d* ≤ 1 and 0 ≤ *c* ≤ *d*. The two contact points *A* and *B* between the LB sofa and inner hallway walls had coordinates of [±(*d*-*c*)·cot(θ/2), -(*d*-*c*)], respectively. Then, we carved out a circular arc (*AB*) passing *O* as the inner border of the LB sofa. On the other hand, the outer borders of the LB sofa were carved out by sweeping out circular arcs (*EG*) and (*FH*) from points *A* and *B*, respectively, so that they are tangent with the outer hallway walls. Finally, critical circular arcs (*AC*) and (*BD*) were formed by sweeping out from the intersection points *E* and *F* to be tangent at points *C* and *D*, respectively, with the lower edge of the unit band of LB. Note that the circular arcs (*AC*), (*BD*), (*EG*) and (*FH*) all have unit radii to make the tangent touches happen simultaneously.

The LB sofa area for the r_i = 0 case $A_{0LB}(\theta)$ is given by the blue outline *ACGEFHDB*, which depends on the parameters *c*, *d*, and θ (**Figure 3A**). Thus, we have:

$$A_{0LB}(c,d,\theta) = A_1 + A_2 + A_3 + A_4 - A_5 \qquad \text{(Equation 1)}$$

Where A_1 is composed of the two circular sectors *AEG* and *BFH*, A_2 is the trapezoid *AEFB*, A_3 the two triangles *ACG* and *BDH*, A_4 the two narrow circular segments *AC* and *BD*, and A_5 the middle circular cutout *AOB* (**Figure 3A**). Note that A_1, A_2, A_3, A_4, and A_5 are, in general, functions of *c*, *d*, and θ. We can write out their equations as follows:

$$A_1(c) = \sin^{-1}(1-c) + \sin^{-1}(c) \qquad \text{(Equation 2)}$$

$$A_2(c,d,\theta) = (1-c)\left[\sqrt{2c-c^2} + 2(d-c)\cot\left(\frac{\theta}{2}\right)\right] \qquad \text{(Equation 3)}$$

$$A_3(c) = c\left(\sqrt{1-c^2} - \sqrt{2c-c^2}\right) \qquad \text{(Equation 4)}$$

$$A_4(c) = \cos^{-1}(1-c) - \sqrt{2c-c^2} \qquad \text{(Equation 5)}$$

$$A_5(c,d,\theta) = (d-c)^2\left[\frac{\theta - \sin\theta\cos\theta}{(1-\cos\theta)^2}\right] \qquad \text{(Equation 6)}$$

The UB sofa area for the r_i = 0 case $A_{0UB}(\theta)$ is given by the red outline of the two parallelograms connected at point *O*, which is contained in the unit band of UB (the red dashed lines in **Figure 3**). $A_{0UB}(\theta)$ depends on θ only:

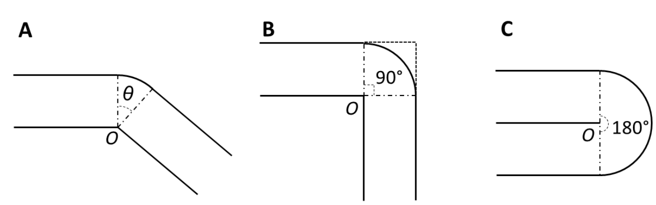

Figure 2: Three diagrams of circular hallways with sharp inner corners. (A) The circular hallway with inner radius r_i = 0 and outer radius r_o = 1 with an arbitrary turn angle θ. (B) The special example when θ = 90°, which is the same as the original moving sofa problem except for the circular outer corner in our case instead of the straight corner in the original moving sofa problem (dashed lines). (C) The special example when θ = 180°, which is a sharp U-turn shape.

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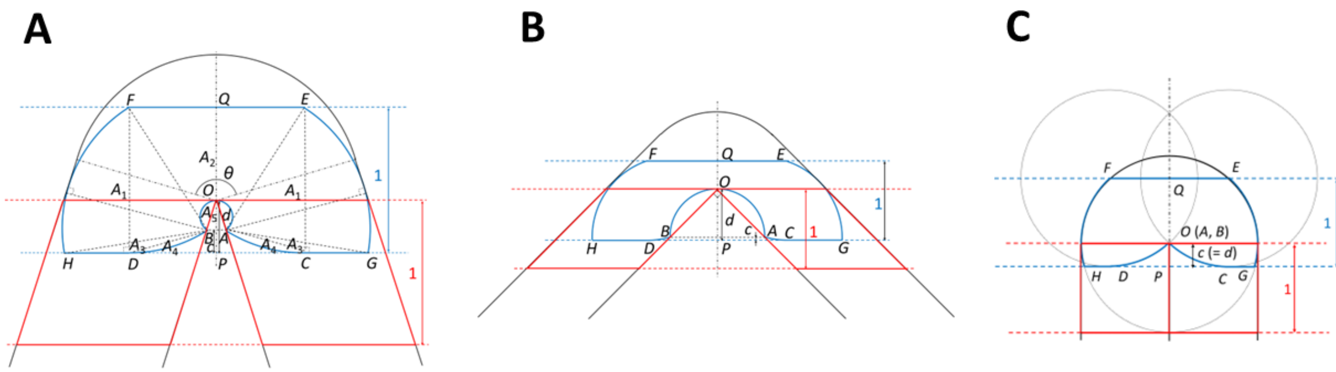

Figure 3: Geometric models for the sharp inner corner problems. (A) The general models to optimize the lower bound (LB) sofa areas that can be moved in a circular hallway with sharp inner corner. The LB model is given by the blue outline *ACGEFHDB*. The two independent parameters *c* and *d* are used to optimize the LB using Desmos at each turn angle *θ*. The upper bound (UB) model is given by the two red parallelograms connected at point *O*, which is a function of *θ* without any optimizable parameter. (B) The LB model (blue outline) and UB model (red outline) formed when *θ* = 90°. (C) The LB model (blue outline) and UB model (red outline) formed when *θ* = 180°.

$$A_{OUB}(\theta) = 2\csc\left(\frac{\theta}{2}\right) \qquad \text{(Equation 7)}$$

As the special examples, we also sketched out the LB and UB geometric models when *θ* = 90° and *θ* = 180°, respectively (**Figure 3B**, **3C**).

We determined the LB and UB of the maximal moving sofa areas from 0° to 180° at a step of 5° **Figure 4**). At each angle, we found the maximum intersection area of the circular hallway of that angle with an infinite band of unit width as either LB or UB of a sofa. We carved out certain portions of the intersection area when calculating the maximal LB to ensure the LB sofa can pass through the circular hallway.

Inner radius	Turn angle θ_{min}(LB) (°)	Lower bound (LB)	Turn angle θ_{min}(UB) (°)	Upper bound (UB)
0.2441	180	1.9899		
0.4142	128.7216	2.1738	180	2.3013
0.5	117.0882	2.2638	171.5463	2.3810
0.75	97.2029	2.5113	154.3324	2.6090
1	**85.8001**	**2.7391**	142.1066	2.8230
1.25	77.9596	2.9502	132.5768	3.0235
1.5	72.0541	3.1474	124.8134	3.2121
1.75	67.3601	3.3331	118.3077	3.3906
2	63.4949	3.5089	112.7433	3.5604
2.25	60.2318	3.6761	107.9084	3.7227
2.5	57.4248	3.8359	103.6538	3.8782
2.75	54.9748	3.9892	99.8706	4.0278
3	52.8109	4.1367	96.4768	4.1721
4	46.1418	4.6793	85.7171	4.7054
5	41.4694	5.1642	77.9147	5.1844

Table 1: Lower bound and upper bound moving sofa areas for the circular inner corner problems after optimization. The turn angles of θ_{min}(LB) and θ_{min}(UB) here are the applicable boundaries of our LB or UB models, respectively. Above such limits, our geometric models yield the same LB and UB results at each inner radius. Note the two limits for inner radius r_i(LB) < 0.2441 and r_i(UB) < 0.4142, respectively. Also note that a special example for r_i = 1 and θ_{min}(LB) = 85.8001° is discussed with the LB determination only and the result is bold highlighted in the table.

However, we did not do so when calculating the UB. Thus, the determined locations of the unit bands of LB and UB are usually different at each angle (**Figure 3**).

When the turn angle *θ* approaches 0°, the maximal moving sofa areas increase sharply from the finite values to infinity. At the high end of the turn angles when *θ* approaches 180°, however, the maximal moving sofa areas decrease much slower. The LB is 1.7352 while the UB is 2 when *θ* = 180° (**Figure 4**). For the special example when *θ* = 90°, our geometric models yielded maximal areas of 2.2153 and of 2.8284 for the LB and the UB sofas, respectively. We added Hammersley's sofa area and the best-known LB result of Gerver's sofa area in our curve plots at *θ* = 90° (3, 4) (**Figure 4**).

Circular Inner Corner Problems

In the previous case, the inner hallway walls consist of two rays intersecting at point *O* (**Figure 2A**). In this more generalized case, we replaced the point *O* with a circular arc of a finite radius centered at *O*. A unit-width circular hallway was plotted with inner radius $r_i = r$ (thus outer radius $r_o = r + 1$) and with an arbitrary turn angle 0 < *θ* ≤ 180° (**Figure 5**).

To estimate the LB sofa area for the case of the circular inner corner, we considered only the situations where the inner radius and turn angle of the circular hallway are both large enough, such that unit band of LB crosses only the circular portion of the hallway (**Figure 6A**). Our geometric model could still make accurate predictions for the LB when outer wall of the straight hallway is in the unit band of LB. When $\theta \geq \theta_{min}$(LB), our model yielded the same value for all turn angles. However, when $\theta < \theta_{min}$(LB), the inner walls of straight hallway enters into the unit band of LB, our model underestimated LB sofa areas. When the inner radius was too small (r_i(LB) < 0.2441), it was impossible for the unit band of LB to dodge the straight inner hallway wall even when the turn angel *θ* reaches 180° (**Table 1**). In this work, we did not look for the more accurate models for the cases where either *r* or *θ* is too small but nonzero.

Particularly, the LB model was formed by sweeping out the inner corners of the LB sofa with two critical circular arcs (*AC*) and (*BD*) such that points on the circular arcs were a unit distance from the upper corners *E* and *F*, respectively. Then the LB sofa was sketched using blue outline *ACGEFHDB*,

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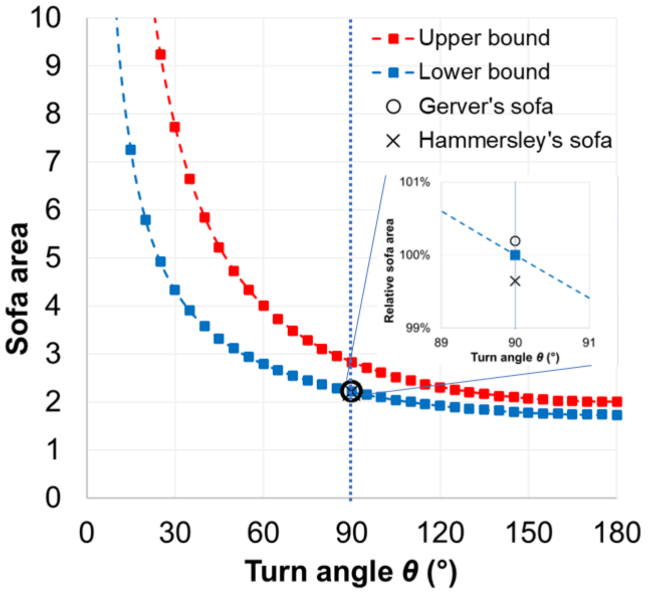

Figure 4: Results of the sharp inner corner problems. The optimized LB and UB of the moving sofa areas that can be moved in a circular hallway with inner radius $r_i = 0$ and outer radius $r_o = 1$ that spans across an arbitrary angle θ. Both the LB of Hammersley's (cross) and Gerver's (open circle) sofa areas are indicated at the blue dotted line $\theta = 90°$ with an enlarged view for the relative sofa area comparison.

assuming the sofa was halfway (**Figure 6A**).

The above steps enabled the inner hallway wall to remain tangent to both critical circular arcs (*AC*) and (*BD*), allowing the LB sofa to be moved through the hallway. In such case, the only optimizable variable was *d*, i.e., the distance of point *O* to the lower edge of the unit band of LB. From the above discussion, we had the requirement $d \geq r \cdot \cos(\theta_{min}(LB)/2)$. We determined the value of *d* for each inner radius *r* that yielded the maximal LB sofa area, while keeping θ large enough. Once we found the optimized value *d*, we calculated the minimum angle θ_{min}(LB) that satisfies the above inequality. Our resulting model will then predict the LB for $\theta \geq \theta_{min}$(LB) by the blue outline *ACGEFHDB*, which depends on both the inner radius *r* and the parameter *d* (**Figure 6A**). Thus, we have:

$$A_{rLB}(r, d) = A_{r1} - A_{r2} \qquad \text{(Equation 8)}$$

Where A_{r1} is the area of the intersection *AICGEFHDJB*, and A_{r2} is the area of the two small cuts *AIC* and *BJD* at the inner bottom of LB sofa. We also have:

$$A_{r1}(r, d) = 2\int_{d}^{1+d} \sqrt{(r+1)^2 - t^2}\,dt - 2\int_{d}^{r} \sqrt{r^2 - t^2}\,dt \qquad \text{(Equation 9)}$$

$$A_{r2}(r, d) = 2\int_{0}^{\sqrt{1-(d+1)^2/(r+1)^2}} \left(1 - \sqrt{1 - t^2}\right) dt - 2\int_{\sqrt{r^2-d^2}}^{r\sqrt{1-(d+1)^2/(r+1)^2}} \left(\sqrt{r^2 - t^2} - d\right) dt \qquad \text{(Equation 10)}$$

A special example, when the inner radius $r_i = 1$ and the outer radius $r_o = 2$, the smallest turn angle θ_{min}(LB) was found to be 85.8001° and the optimized LB sofa area was *A*1LB = 2.7391 (**Table 1**, **Figure 3B**). Exactly at the minimal turn angle θ_{min}(LB) = 85.8001°, the lower edge of the unit band

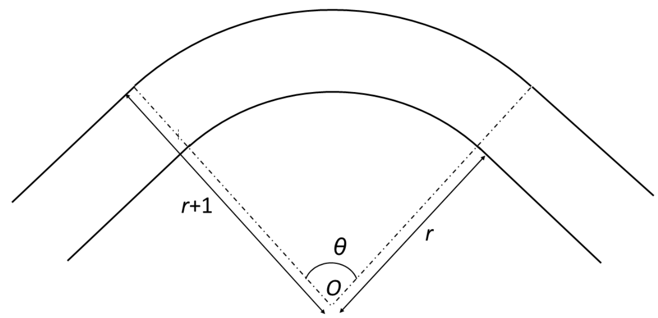

Figure 5: A hallway with a circular inner corner. The hallway includes left straight portion, right straight portion, and a circular ring with inner radius $r_i = r$ and outer radius $r_o = r + 1$ to connect them across an arbitrary turn angle θ. The point *O* is located at the center of the ring-shaped hallway.

of LB passes the two intersect points of circular and straight portions of the inner hallway wall after model optimization (**Figure 6B**). Below the turn angle limit, our geometric model is not accurate anymore due to the additional unused room in the straight portion of the hallway.

Our UB model was given by the intersection of the unit band of UB with the circular hallway (red outline; **Figure 6A**). Its area was given by:

$$A_{rUB}(r, d') = 2\int_{d'}^{1+d'} \sqrt{(r+1)^2 - t^2}\,dt - 2\int_{d'}^{r} \sqrt{r^2 - t^2}\,dt \qquad \text{(Equation 11)}$$

Where *d'*, i.e., the distance of point *O* to the lower edge of the unit band of UB, is an independent variable (relative to *d*) when determining the UB. Typically, *d'* is smaller than *d* after optimization, thus the UB model is below the LB model in the same hallway (**Figure 6A**). The fact that $d' < d$ is understandable because for the UB model, we did not deduct the two inner bottom areas. The area optimization procedure made *d'* smaller than *d*, resulting in a wider and larger UB sofa than a LB sofa (**Figure 6A**).

For the UB estimations, a similar limit appeared when the inner radius is too small (r_i(UB) < 0.4142), below which it is impossible for the unit band of UB to dodge the straight inner hallway wall even when the turn angle θ reaches 180° (**Table 1**). Also, we considered only the situations where the turn angle θ was large enough so that the unit band of UB crossed only the circular portion of the hallway. Our UB model could not make accurate predictions when either straight outer walls or straight inner walls enter the unit band of UB. Thus, in this work, our UB models yielded effective results when $\theta \geq \theta_{min}$(UB) only, where θ_{min}(UB) is typically larger than θ_{min}(LB) at the same inner radius (**Table 1**).

The LB and UB of maximal moving sofa areas were listed at different inner radii (**Table 1**). As in the above discussion, our effective modeling started with a limit of inner radius r_i(LB) ≥ 0.2441 for the LB and with a limit of r_i(UB) ≥ 0.4142 for the UB. The table listed out the minimal turn angles θ_{min}(LB) and θ_{min}(UB) at each inner radius as well. When the turn angle $\theta \geq \theta_{min}$(LB) or $\theta \geq \theta_{min}$(UB), the respective LB and UB values were applied. The LB and UB values were also plotted as 3D curves at such minimal turn angles θ_{min}(LB) and θ_{min}(UB), respectively, at the different inner radii (**Figure 7**). Both the LB and UB of the moving sofa areas increase monotonically with the inner radii of the circular hallway and decrease

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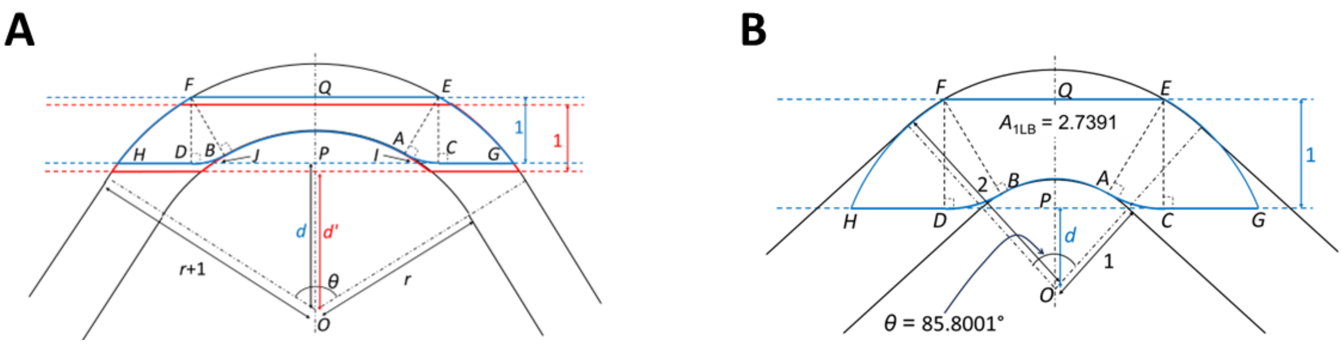

Figure 6: Geometric models for the circular inner corner problems. (A) The general models to optimize the LB sofa areas that can be moved in a circular hallway with inner radius r_i = r and outer radius $r_o = r + 1$. Note that the LB model (blue outline *ACGEFHDB*) has circular cuts at the lower bottom inner corners, while the UB model (red outline) does not have such cuts. Using Desmos at each inner radius *r*, the parameter *d* is used to optimize the LB, while the parameter *d*' is used to optimize the UB. (B) The optimized LB model for a special example of the circular hallway with inner radius r_i = 1 (outer radius r_o = 2) and a minimal turn angle θ_{min}(LB) = 85.8001°. The optimized LB area A_{1LB} is 2.7391. Note that the lower edge of the unit band of LB passes the two intersect points of circular and straight portions of the inner hallway walls. When the turn angle is even smaller, the straight inner hallway walls enter the unit band of LB, and our LB geometric model will be not applicable anymore.

monotonically with the turn angles (**Table 1, Figure 7**).

DISCUSSION

By using the proposed geometric methods, we calculated the LB and UB of the moving sofas that can be moved through a unit-width circular hallway with an arbitrary turn angle θ (0° < θ ≤ 180°). We allowed the inner radius of such a circular hallway to be either zero or above zero and estimated the LB and UB of the moving sofas.

For the special case of the inner radius r_i = 0 and turn angle θ = 90°, which resembled the original moving sofa problem, our LB area of 2.2153 matched quite well with the best-known Gerver's sofa area of 2.2195 with a ~0.2% difference (**Figure 4**). Such an excellent match not only reconciled the effectiveness of our methods used in the study, but also confirmed our third hypothesis. Since Hammersley's and Gerver's sofas have circular curves at the out corners, we expected that these sofas can move through our defined 90° circular hallway as well. Hammersley's sofa turned out to be a special case in our geometric modeling by setting the two parameters to be *c* = 0 and *d* = 2/π ≈ 0.6336. While our two independently optimizable parameter modeling yielded a ~0.3% better result with *c* = 0.0338 and *d* = 0.6483 (**Figure 3B, 4**). Although Gerver's sofa still fits through the corner of a 90° circular hallway, due to changing tangential restrictions to optimize, the navigation path is more complicated than that of Hammersley's LB sofa and that of our LB sofa (7).

Our new results may have potential applications in areas like design of plumbing systems for less clogging and easier cleaning, or design of specialized vehicles to be moved through existing circular tube systems. For example, the 180° sharp U-turn is often used in low-profile compact heatsink systems for returning water flow (**Figure 2C**). We can design a mechanical part with the LB shape (blue outline) for efficient residue/clog removal (**Figure 3C**).

In future work, we will seek to eliminate the limitations on both inner radii and turn angles so that the moving sofa area $A(r, \theta)$ can be determined for any $r_i \geq 0$ and any arbitrary angle 0° < θ ≤ 180°. Analytical methods like those developed for the original moving sofa problem can be useful to further improve the LBs and UBs (4, 6, 7, 10-12). Also, since the scope of 2D motion planning is more limited, it may be more engineeringly useful to extend 3D plumbing systems with more arbitrary turn angles (13-16). Finally, while the extended moving sofa problems provide a theoretical framework, real-world applications often involve additional complexities like uneven surfaces, obstacles, and dynamic environments, requiring further adaptation and refinement of the concept. Solving the various moving sofa problems for real-time complex shapes can be computationally expensive, requiring various optimization techniques to make it practical in

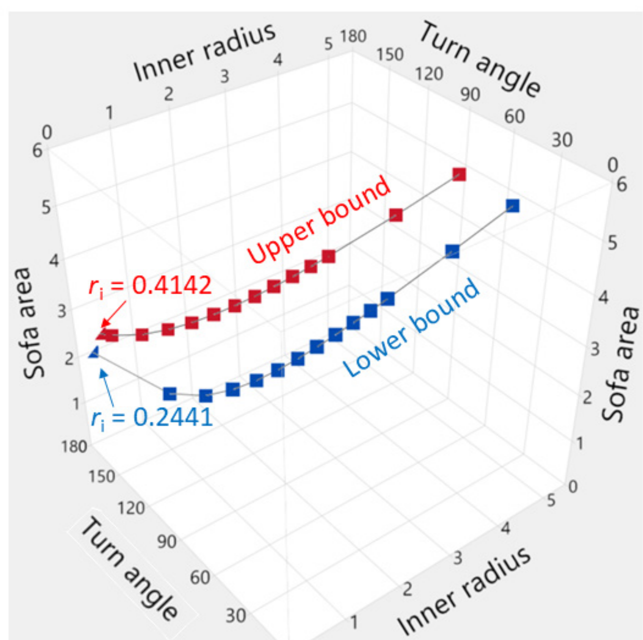

Figure 7: Results of the circular inner corner problems. The optimized LB (blue squares) and UB (red squares) of the moving sofas can be moved in a circular hallway with a turn angle $\theta \geq \theta_{min}$(LB) and $\theta \geq \theta_{min}$(UB), respectively, at the different inner radii. Our modeling results start with r_i(LB) ≥ 0.2441 for the LBs and with r_i(UB) ≥ 0.4142 for the UBs.

certain applications. In terms of such practical concerns and tradeoffs for real-world applications, our geometric methods may provide a better balance between computation costs and closer estimations than other more complicated methods.

MATERIALS AND METHODS

Like in the standard moving sofa problem, the equality of the rotation angle of the moving sofa with the turn angle θ of the circular hallway was assumed, and the sofa was assumed to be symmetrical. Again, since the moving sofa must fit in both the straight and the circular turning portions of the hallway, the intersection of the hallway with a unit-width band, which is the unit band of LB or that of UB, was considered. To construct the geometric model, the moving sofa was confined in the intersection of the unit band and the hallway. The intersection was a single connected region as defined by our problem statement.

To determine the UB, the intersection area of the unit band of UB and the hallway was maximized assuming the UB sofa would be at the halfway. For the case of sharp inner corners, such maximal intersection area would be two parallelograms connected at point *O*. For the case of the circular inner corner, the maximum intersection area was all contained in the circular hallway portion with an optimizable parameter *d'* (**Figure 3A**, **6A**).

To determine the LB, all the pieces that needed to be carved out from the intersection of the unit band of LB and the hallway were identified to ensure the LB sofa can fit through. Then independent parameters to determine the LB for the new situation were optimized. For the case of sharp inner corners, such a modified intersection area was found to be a function with two optimizable parameters *c* and *d* (Equation 1), while for the case of circular inner corners, such a modified intersection area was found to be a function with one optimizable parameter *d* only (**Equation 8**, **Figure 3A**, **6A**).

Once geometric models were established and optimizable parameters were identified, the online calculator Desmos was used to maximize the LB and the UB of the moving sofa areas and to find the optimal parameters at each turn angle and inner radius.

In the following two paragraphs, we showed in detail how the LB sofa fits through the unit-width hallway for the case of sharp inner corner.

Let points *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H* be the fixed points on the LB sofa borders and point *O* be the hallway's sharp inner corner (**Figure 3A**). Initially, the LB sofa was all within the left straight portion and touching the left straight hallway walls with straight edges of *GC*, *DH*, and *EF*. Then the LB sofa was pushed through without any rotation until *C* and *O* coincide. We then pushed and rotated the LB sofa until the inner corner point *A* approached *O*. At this stage, the LB sofa edge circular arc (*BD*) remained tangent to the left inner hallway wall. Since all the points on circular arc (*EG*) were less or equal to one unit distance from circular arc (*AC*), and point *F* was always one unit distance away from circular arc (*BD*), the outer hallway wall would not cut through the defined LB sofa area during the rotation. Once the LB sofa inner corner *A* passed point *O*, we rotated the sofa more aggressively with point *A* constantly touching the right inner hallway wall, while the left inner hallway wall remained tangent to circular arc (*BD*). Such rotational movement continued until the left inner hallway wall became perpendicular to line *FB*, at which point the left inner wall could not remain tangent to circular arc (*BD*) any longer. Since the inner hallway wall now touched both points *A* and *B*, it implied that *O* also began to touch the inner border of the LB sofa circular arc (*AB*). At the last stage, we continued to rotate the LB sofa until it reached the halfway position (**Figure 6A**). Once the LB sofa finished rotation by $\theta/2$, the above steps could be duplicated into the second half stages due to the symmetry of both the hallway and the sofa.

At all the above-described stages, there were always some points on the critical circular arcs (*AC*) and (*BD)* of the LB sofa edges touching the inner hallway wall, either its left or right portions. This ensured that the outer edges of the LB sofa could fit through the outer hallway wall, which were always one unit away from the inner wall. Then, what remained was to vary the parameters *c* and *d* for each turn angle θ to get the optimal LB area of the moving sofa.

Compared to the case of hallway with sharp inner corner, the main difference of our geometric modeling for the case of circular inner corner was that only one parameter *d* needs to be optimized. In such case, the inner middle circular cutout of the modeled sofas followed exactly the inner circular hallway wall.

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Appendix

Assessable online resources for the paper:

1) Sofa area finder for the Sharp Inner Corner Problems: https://www.desmos.com/calculator/tqoqvud6jt
2) Sofa area finder for the Circular Inner Corner Problems: https://www.desmos.com/calculator/ve5iwtoumo