

Estimating the liquid jet breakdown height using dimensional analysis with experimental evidence

Rusu Paisie¹, Cârlig Sergiu²,

¹ Aristotel” Theoretical Lyceum, Chişinău, Republic of Moldova

² Institute of Applied Physics, Chişinău, Republic of Moldova

SUMMARY

When opening a sink or pouring water from a bottle an interesting and unintuitive physical phenomenon occurs: the regular cylindrical jet breaks into droplets at a certain height, which changes once the initial conditions of the falling jet change, thus two distinct regions are observable: a steady stream of water at high flow and separate droplets at low flow. In this paper, we found and experimentally tested a model that correlates the parameters of the phenomenon mentioned above. We analyzed how the breakdown height of the jet changes with three initial parameters: jet radius, liquid density, and surface tension. We propose a mathematical model for the relationship between the diameter of the liquid stream and the height at which it breaks into droplets, consisting of a quadratic dependency. There was a good fit between experimental data and the model with a correlation coefficient exceeding 0.95. Finding and exploring this way to analyze fluid mechanics might help understand the fundamental correlation between the parameters involved and design more lucrative industrial techniques in fields such as oil extraction or printing.

INTRODUCTION

Drop formation in a falling jet of liquid is an interesting and often observed hydrodynamic phenomenon, derivations of which are commonly used in industrial applications, such as fuel injection, ink-jet printers, and fiber spinning (1,2). This phenomenon was first described by Rayleigh, who discovered in the 19th century that surface tension is the driving force in the drop formation, as it tends to minimize the surface area of the jet, turning it from a cylinder to a sphere (1). The surface tension forces drive Rayleigh-Plateau instability: If the radius of the jet decreases, surface tension forces induce a local increase in fluid pressure, and conversely, a local increase in radius causes a local decrease in pressure. Thus, fluid will flow from regions of depression to regions of expansion, and according to the principle of mass conservation, the regions of expansion will increase (8). This instability begins with the existence of minor irregularities in the jet. These are always present, no matter how smooth the stream looks. Vibrations can induce them due to friction between the pipe and the liquid stream (9).

Studies have revealed several factors that influence the length a water stream can maintain continuously after leaving

the container: liquid viscosity, flow rate, the ratio of capillary internal and external diameter of the recipient, and surfactant distribution (the distribution of substances such as a detergent that, when added to a liquid, reduces its surface tension, thereby increasing its spreading and wetting properties) (10).

More explicitly, the problem this paper focused on consists of the following: a jet of a Newtonian fluid, initially of constant radius, is falling vertically under gravity. The liquid length increases and reaches a critical value. At this critical value, the jet loses its cylindrical shape as it decomposes into a stream of droplets (Figure 1) (4).

The Equation for the height at which a jet breaks into droplets has been derived using the Navier-Stokes equations. However, it cannot be defined analytically with basic algebraic operations and functions, instead relying on modified Bessel function as one of its factors (2). This represents a problem when an approximation of the model is needed to reduce computational complexity. The goal of this study was therefore to develop a model of the jet stream that uses simple algebraic equations.

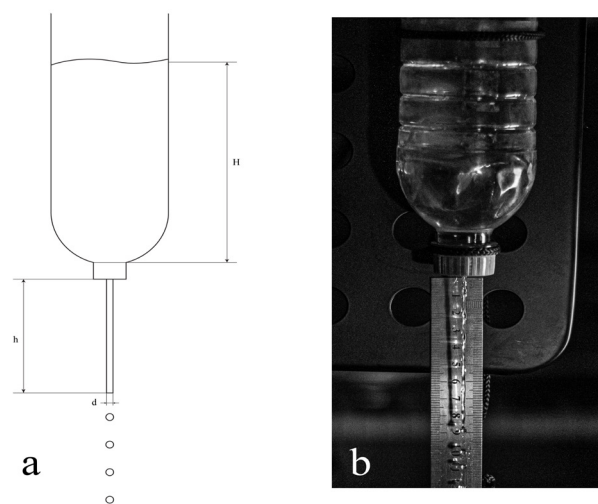


Figure 1. Diagram of the main experimental setup. The illustration (a) and image (b) show an upside-down plastic bottle with a hole in the cap, allowing a jet of water to form. The three measured values are h , the distance between the cap of the bottle and the point at which the jet turns into separate droplets; d , the diameter of the jet (approximated to be constant, equals to radius times two); H , the height of the column of water above the cap.

Tables 1.1, 1.2 Experimental data points for the radius of the falling jet(R), breakdown height, and water column height. The table shows the measured radius (R= diameter divided by 2), the corresponding height *h* at which the jet breaks into individual droplets, the water column height *H*, standard deviation, and relative error, and the values $R^{(3/2)}$. The setup (figure 1) was photographed with an exposure time of 1/3200 s then data was collected manually for each entry.

R. (m)	h. (m)	H. (m)	ΔH (m)	ε _H (%)	R ^(3/2) , (m ^(3/2))
0,00030	0,020	0,1236	0,0009	0,78	0,00000527
0,00083	0,090	0,1225	0,002	1,70	0,00002405
0,00097	0,106	0,1282	0,0035	2,80	0,00003041
0,00089	0,073	0,1237	0,0008	0,67	0,00002683
0,00155	0,180	0,1254	0,0008	0,68	0,00006115
0,00124	0,117	0,1275	0,0029	2,31	0,00004383
0,00116	0,142	0,1226	0,0019	1,62	0,00003954
0,00118	0,141	0,1209	0,0036	3,02	0,00004089
0,00159	0,297	0,1240	0,0005	0,42	0,00006351
0,00207	0,282	0,1246	0	0	0,00009465
0,00242	0,489	0,1272	0,0026	2,09	0,00011936
Average:		0,1246	0,0018	1,46	0,00004995

R. (m)	h. (m)	H. (m)	ΔH (m)	ε _H (%)	R ^(3/2) , (m ^(3/2))
0,00035	0,018	0,1043	0,0021	2,01	0,00000658
0,00085	0,085	0,1055	0,0008	0,82	0,00002482
0,00085	0,092	0,1062	0,0001	0,11	0,00002509
0,00093	0,085	0,1065	0,0001	0,09	0,00002837
0,00117	0,132	0,1070	0,0006	0,60	0,00004035
0,00085	0,072	0,1071	0,0007	0,68	0,00002509
0,00128	0,129	0,1097	0,0033	3,01	0,00004610
0,00111	0,12	0,1055	0,0008	0,80	0,00003703
0,00207	0,33	0,1053	0,0010	1,03	0,00009448
0,00213	0,368	0,1068	0,0004	0,38	0,00009874
Average:		0,1064	0,0010	0,95	0,00004266

In this paper, we focused on exploring the algebraic correlation between the measurable parameters of the jet: radius, density, surface tension, and the time and height at which said jet would break into droplets. We brought experimental evidence to support the proposed correlation between breakdown height and jet radius. This relationship could prove helpful in industrial and laboratory applications, as it could predict the critical time and height (critical denoting the moment when the jet splits into droplets), knowing only the measurable properties. The breakdown height is closely tied to the critical time of transformation into droplets. Finding ways to express this dimension is essential for advancing our intuition and understanding of the Rayleigh-Plateau instability.

After bringing experimental data to support our claims, we found that our model has a reasonable degree of accuracy while being relatively simple from a mathematical standpoint.

RESULTS

Model

Starting from the assumption that the breakdown time *T* is a function of the jet radius *R*, the surface tension coefficient σ and the water density ρ , so we can write

$$T = c \cdot \rho^a \cdot R^b \cdot \sigma^d$$

Since the unit of *T* has to be seconds, we use the method of dimensional analysis to find the smallest values of the exponents as $a=0.5$; $b=1.5$; $d= -0.5$, and we write:

$$T = c \sqrt{\frac{\rho R^3}{\sigma}} \quad (1)$$

Where *c* is an arbitrary constant, as the method of dimensional analysis could only indicate the proportionality between factors, which does not change when multiplying with a constant. This equation has been obtained similarly by Oren Breslouer *et al*. We plugged Equation (1) into the equation of the falling body $h = v_0 t + c \sqrt{(at^2/2)}$, where *t* is substituted by Equation (1) and $a = g$, the gravitational acceleration, to obtain the jet breakdown length,

$$h = v_0 c \sqrt{\frac{\rho}{\sigma} R^3} + \frac{gc^2 \rho}{2\sigma} R^3 \quad (2)$$

Where v_0 is the velocity of the jet as it exits the bottle. The value of *c* was determined using the least squares method to suit the graph best, as this is the most accurate analytical method of determining the value of an unknown coefficient⁶.

During this analysis, we assumed the jet radius *R* to be constant, which is not the case. According to the continuity law (the flow of mass is constant in a steady stream), the radius *R* varies with the height *h* as

$$R = R_0 \sqrt[4]{\frac{v_0^2}{v_0^2 + 2gh}}$$

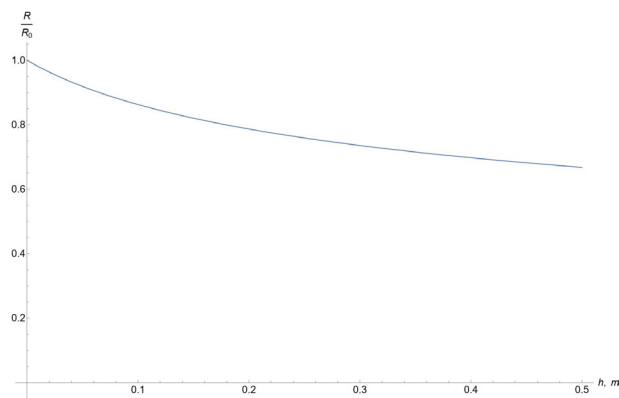


Figure 2. Plotted dependence of jet radius versus height. The line represents the dependence of the jet radius (assumed constant in the model) on the jet height (R/R_0) – ratio of the jet radius at a certain height to the initial jet radius, h – the height, m – meters). The data was collected so that the assumption would guarantee an error of less than 15 %.

Where R_0 is the initial radius and v_0 the initial velocity of the falling jet (Figure 2)¹. We measured the radius of the jet at the height of 2 – 5 cm, and the jet started dispersing into droplets at an average height of 12 – 20 cm. In the cases where the breakdown height was larger than average, we measured the radius lower. Considering these factors and the dependence of the radius jet on the jet height (Figure 2), the radius's relative difference should not exceed 15 % making our approximation acceptable.

To experimentally test our hypothesis, we created a setup using a plastic bottle, rulers, and caps with holes of different diameters on them (Figure 1). We measured the height at which the water cylinder split for multiple values of the jet diameter and two values of the water column height H (Table 1.1, 1.2). This experiment allowed us to verify the relation between the critical length and jet radius. We made two sets of measurements with different water column heights and, thus, different initial velocities. We plotted the modified quadratic function $h(R)$ defined in Equation (2), after determining the value of the coefficients using the least squares methods (we found the curve that minimizes the sum of distances between the experimental points and the line) ($c_1 \approx 13.237$, $c_2 \approx 13.726$), and experimental points (Figure 3). After performing an error analysis, we determined the correlation coefficients between our model and the experimental data were $r_{c1} = 0.95$ and $r_{c2} = 0.99$ (7).

DISCUSSION

We hypothesized that the critical time scale of the jet breakdown would depend on the radius of the jet and the density and surface tension of the liquid. Using dimensional analysis, we deduced that a theoretically acceptable dependence would be Equation (1), and we wrote Equation (2)

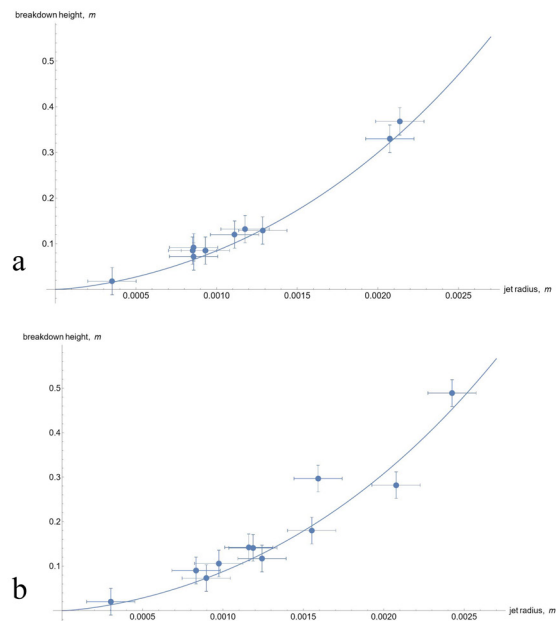


Figure 3. Plotted dependences of height and diameter – theoretical vs. experimental. Figure 3a is for the lower water column height, while Figure 3b is for the higher water column height. The line represents the graph of Equation (2); the points are the experimentally obtained data points. We used the average c -values presented above to test the single-variable model.

following the fundamental laws of the falling body. Then, we experimentally tested the radius-height dependence, plotted the theoretically obtained function and the experimental data points, and computed the correlation factor⁷. This factor should be zero for totally unrelated values and one for a perfect correlation. The minimum (worst case scenario) correlation factor for our experiment was above 0.95, suggesting a very tight correlation between our proposed model and the experiment. We found the line of best fit using the least square method⁶.

There are several limitations in our experiment and the model. It has been shown in the literature that the critical length also depends on the wavelengths of microscopic disturbances induced in the jet when it leaves the bottle⁵. Without these disturbances, or another weak point, the jet would transform into a big spheroid droplet. The impact of these wavelengths is incorporated into the constant c - we assumed them to behave in the same way for the entire experiment. The evidence suggests they did (a correlation coefficient higher than 0.8 suggests a good fit between the variables; in our case, this factor was above 0.95. We used the average value of c when computing it).

A technical limitation of the experiment is the possible range of jet radius values. If we could get data points for

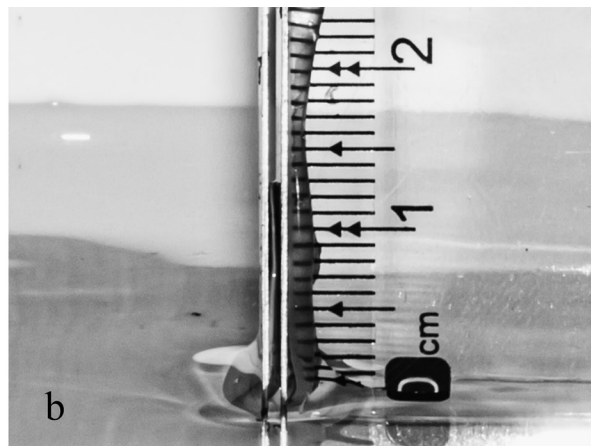
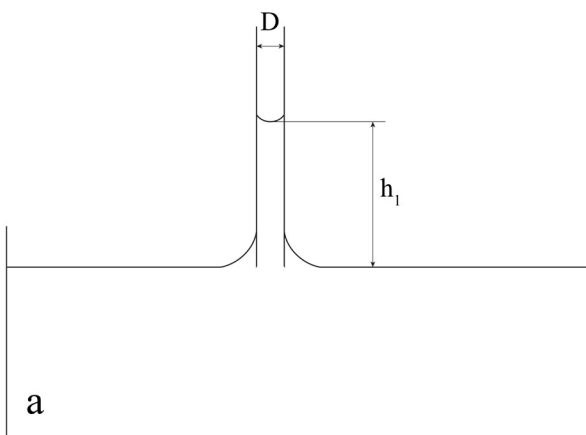


Figure 4. The setup used to measure the surface tension of water Two blades with a small gap D in between are placed in a recipient of water. The water rises due to surface tension up to a height h_r . (a) shows an illustration, while (b) shows a setup photo.

a radius smaller than 0.3 mm and more than 2.2 mm our confidence in the correctness of the model would be higher. Per the continuity law, we also did not consider the jet radius variation, which decreases as the jet speed increases. This model would allow a significant simplification in calculations and industry processes. However, our model is not that versatile and would require much more theoretical and practical work to become reliably used, especially for parameter values outside of the measured range. Our results are on par with those of similar studies . While we

experimentally supported Oren Breslouer's proportionality (which he supported numerically), the constant is different, confirming its relationship with the specific wavelengths and, thus, dependence on the mechanical properties of the setup⁴. The reliability of such a simple algebraic approach in fluid mechanics could improve the intuition and ability of researchers to make a rough estimation of the parameters of the phenomenon

The field of fluid mechanics has become increasingly more studied in recent years with the advancements of

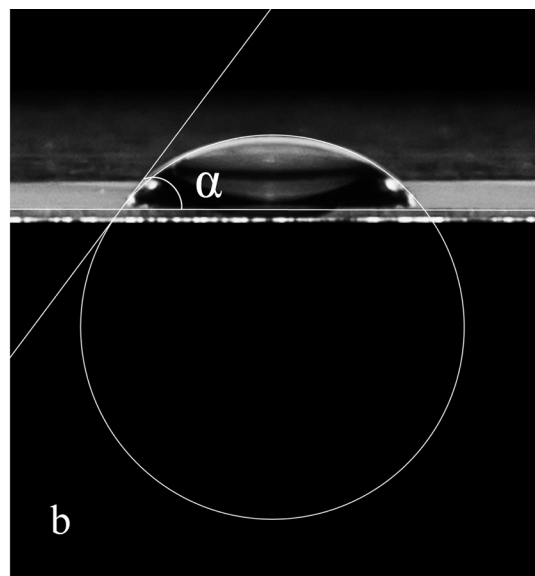
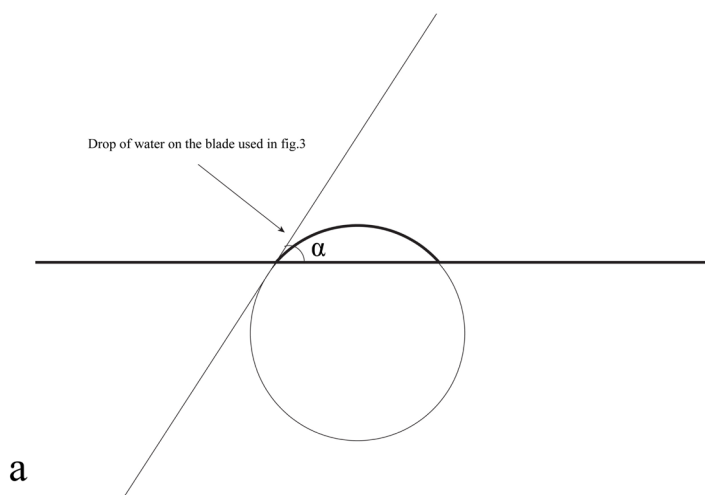


Figure 5. The setup used to measure the angle of adhesion between the water and the surface. A small drop of water is put on one of the identical blades used in Figure 3. Angle α of adhesion is measured by drawing an imaginary circle around the droplet and its tangent at the point of intersection with the blade. (a) shows an illustration, while (b) shows the actual setup.

computer simulation techniques and other modern ways of research. That said, there are some novel research ideas, like a precise, fully analytical model that describes the breakdown of a jet into droplets. To further our investigation, we could analyze the droplet formation in a jet of non-Newtonian fluid, where the viscosity would not be negligible.

We have described a simple, intuitive mathematical model that correlates when and where a falling jet of water breaks down into droplets, a model that is accurately supported by experimental data on a defined set of values. This model is meant to simplify future work in the area and intuitively show the correlations between specific physical parameters. While with some undeniable limitations, our work contributes to the understanding and use of fluid dynamics, mainly the phenomena related to Rayleigh-Plateau instability.

MATERIALS AND METHODS

Main experiment

A 0.75 L plastic bottle with its bottom cut off was suspended upside down on a general-purpose adjustable stand. We fit it with a bottle cap with a round hole cut. Two 30 cm rulers were also suspended, so the bottle cap was precisely at the 0 cm mark. We filled the bottle with water via its cut-out bottom, and the whole setup was photographed with a Canon 600D equipped with a Canon 50 mm f1.8 STM lens (set on manual focus) as the water jet poured out of the cap and broke down into droplets at a certain height. The steps mentioned above were repeated twice for 23 different bottle caps with holes of different radii, using two reference values for H to try to get the initial velocity constant and make it possible to plot them in the same graph with one trendline.

The photos were then uploaded into Adobe Lightroom Classic CC 2021, retouched, cropped, and decolorized (for a better observation of the jet edges). After, we printed on large pieces of paper and measured with rulers the water column height (H), jet breakdown height (h), and jet diameter ($2 \cdot R$). We used the rulers in the image to scale the dimensions, keeping perspective into account. d is the jet diameter, measured as low as possible before the jet begins to turn into droplets in order to minimize the error caused by assuming it is constant, h is the breakdown height or the distance in meters from the bottle cap to where the first droplet breaks from the cylindrical jet, and H is the water column height.

We took the photos continuously. We selected one photo for every jet diameter, with H closer to 12.46 cm and respectively 10.64 cm (values corresponding to distinct "rings" in the bottle design), so that our data is consistent and we can plot in on a common coordinate system. We assumed the jet has a constant radius, which is not factually true. To calculate the error induced by this assumption, the plotted $(R/R_0)(l)$ and calculated the relative difference at the point where we measured the radius and at the breakdown point. No relative error exceeded 15%. The absolute error for the height experimental measurements is no larger than $3 \cdot 10^{-2}$ m and for the diameter – no larger than $1.5 \cdot 10^{-4}$ m (Figure 3).

According to the continuity law, the ratio between the velocities of the water from the jet and the bottle is the ratio of the respective radii squared. As the radius of the bottle is on the order of centimeters and that of the jet is millimeters, the velocity of the water jet would be two orders of magnitude higher. The ratio of the velocities squared would be four orders of magnitude; thus, we safely assumed that the square of the bottled water velocity was 0. According to the Bernoulli law (static pressure + dynamic pressure is constant in a stream), we approximated the speed of the falling jet as Equation (4).

In order to calculate the value of the arbitrary constant c , we got a substitution $R^{(3/2)}=x$, thus Equation (2) became $h(x)=Ax^2 + Bx$ where

$$A = \frac{gc^2\rho}{2\sigma} \quad , \quad B = v_0c\sqrt{\frac{\rho}{\sigma}}$$

Using the method of least squares, the coefficients A and B were determined and then the proportionality constant c^6 .

For the data set with $H = 12,46$ cm the c is equal to 12.197 as expressed from A and 14.276 from B . While our model predicted these values should be equal, we can notice they differ by about 16%. However, this is in line with the error margin of the assumption that the radius is constant and measurement errors. To comply with the proposed single-parameter model, we will use the average value $c = 13.236$ to plot the graphs and calculate the correlation coefficient.

We proceeded similarly for the second data set, and $c_A = 12.826$ and $c_B = 14.625$ were the values found for the constants obtained from A and B , respectively. The median value $c = 13.726$ was the one we used further. We calculated the coefficients of correlation as:

$$r_c = \sqrt{1 - \frac{\Sigma(h_i - h(R_i))^2}{\Sigma(h_{med} - h_i)^2}}$$

Where h_i are the experimentally found heights, $h(R)$ is Equation (2) (with the coefficients computed according to the found c -values, not those resulting from the Least Squares computation), R_i are the measured values of the jet radius and h_{med} is the mean of all measured heights⁷.

Surface tension determination

Surface tension is the tendency of liquid surfaces at rest to shrink into the minimum surface area possible.

We considered the value of the gravitational acceleration to be 9.81 ms^{-2} and the density of distilled water at room temperature - $998 \text{ kg}\cdot\text{m}^{-3}$. We experimentally measured the value of surface tension (Figure 4, 5):

$$\sigma = \frac{\rho D h_1 g}{2 \cos(\alpha)} \quad (3)$$

Where ρ is the water density, D is the distance between the blades, h_r is the height at which the water rises due to surface tension, g is the gravitational acceleration, and α is the angle of adhesion between the surface of the blades and the water. We measured h_r to be $1.29 \cdot 10^{-2}$ m, $D=6.58 \cdot 10^{-4}$ m, $\alpha=56^\circ$, giving us $\sigma=0.0737$ Nm⁻¹.

We approximate the initial velocity of the falling jet as follows:

$$v_0 = \sqrt{2gH}. \quad (4)$$

Where g is the gravitational acceleration and H is the water column height.

We put a droplet of water on an unbranded stainless steel blade and photographed it with the same photo equipment as in the first experiment. We then imported the photo in Adobe Photoshop, drew a circle to comprise the droplet best and measured the angle α of adhesion between the tangent of the circle where it crossed the plane of the blade and the horizontal. We then glued together two of the same blades, putting another two blades in between to make a gap of width D (we precisely measured D by putting together ten blades, measuring the width, and dividing by ten).

Then the above-described contraption was put in water, and we measured the level h_r at which the water rose (**Figure 4**). We did not perform error analysis for this experiment as the obtained value was close to the tabular one (0.075 Nm⁻¹ at 20°C , which means that we are off by about 1.8%), and the precision of σ does not greatly impact the precision of the main experiment.

We can deduce Equation(3) by setting equal the force of gravity acting on the water trapped between the blades and the force of surface tension, which is equal to the product of the surface tension and the perimeter on which it acts and is oriented at an angle α respective to the surface.

ACKNOWLEDGMENTS

I would like to thank "Aristotel" Theoretical Republican Lyceum and physics teacher Sergiu Bîzgan for invaluable help in offering laboratory space, advice, and mentorship during this project.

Received: March 25, 2023

Accepted: June 02, 2023

Published: June 20 2023

REFERENCES

1. Grubelnik, Vladimir, and Marko Marhl. "Drop Formation in a Falling Stream of Liquid." *American Journal of Physics*, vol. 73, no. 5, 2005, pp. 415–419., doi:10.1119/1.1866100.
2. Kowalewski, T A. "On the Separation of Droplets from a Liquid Jet." *Fluid Dynamics Research*, vol. 17, no. 3, 1996, pp. 121–145., doi:10.1016/0169-5983(95)00028-3.
3. Domenichini, F., and B. Baccani. "A Formulation of Navier–Stokes Problem in Cylindrical Coordinates Applied to the 3D Entry Jet in a Duct." *Journal of Computational Physics*, vol. 200, no. 1, 2004, pp. 177–

- 191., doi:10.1016/j.jcp.2004.04.002.
4. Breslouer, Oren. "Rayleigh-Plateau Instability: Falling Jet - Princeton University." Princeton.edu, 1 Aug. 2010, www.princeton.edu.
5. Massalha, Taha, and Rafael M. Digilov. "The Shape Function of a Free-Falling Laminar Jet: Making Use of Bernoulli's Equation." *American Journal of Physics*, vol. 81, no. 10, 2013, pp. 733–737., doi:10.1119/1.4819196.
6. Levenberg, Kenneth. "A method for the solution of certain non-linear problems in least squares." *Quarterly of applied mathematics* 2.2 (1944): 164-168.
7. Asuero, A. G., et al. "The Correlation Coefficient: An Overview." *Critical Reviews in Analytical Chemistry*, vol. 36, no. 1, 2006, pp. 41–59., doi:10.1080/10408340500526766.
8. Papageorgiou, Demetrios T. "On the Breakup of Viscous Liquid Threads." *Physics of Fluids*, vol. 7, no. 7, 1995, pp. 1529–1544., doi:10.1063/1.868540.
9. Eggers, Jens. "Nonlinear Dynamics and Breakup of Free-Surface Flows." *Reviews of Modern Physics*, vol. 69, no. 3, 1997, pp. 865–930., doi:10.1103/revmodphys.69.865.
10. Wang, Tongju, et al. "Droplets Generator: Formation and Control of Main and Satellite Droplets." *Colloids and Surfaces A: Physicochemical and Engineering Aspects*, vol. 558, 2018, pp. 303–312., doi:10.1016/j.colsurfa.2018.08.046.
11. Liu, Shanlin, et al. "The Falling, Necking, and Break-up of Droplet between Horizontal Tubes with Low Reynolds Number." *International Journal of Multiphase Flow*, vol. 142, 2021, p. 103717., doi:10.1016/j.ijmultiphaseflow.2021.103717.
12. .

Copyright: © 2023 Paisie & Sergiu. All JEI articles are distributed under the attribution non-commercial, no derivative license (<http://creativecommons.org/licenses/by-nc-nd/3.0/>). This means that anyone is free to share, copy and distribute an unaltered article for non-commercial purposes provided the original author and source is credited.