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Article

The effect of viscous drag on damped simple harmonic motion

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SUMMARY

Dynamic viscosity is a quantity that describes the magnitude of a fluid's internal friction or thickness. Traditionally, scientists measure this quantity by either calculating the terminal velocity of a falling sphere or the time a liquid takes to flow through a capillary tube. However, they have yet to conduct much research on finding this quantity through viscous damped simple harmonic motion. This phenomenon occurs when the movement of an oscillator experiencing Hooke's law dissipates at a rate defined by the damping coefficient due to a fluid's external drag force. The present study hypothesized that the relationship between the dynamic viscosity and the damping coefficient is positively correlated. To test this hypothesis, we computed the dynamic viscosities of glycerol-water mixtures of varying mass concentrations and filmed the oscillations of a vertical-spring mass system in each of these mixtures. We then determined the system's damping coefficients through Vernier's Logger Pro video analysis feature and implemented inferential statistics on a regression between these damping coefficients and the mixtures' dynamic viscosities. The results through these methods demonstrated not only compelling evidence in favor of this hypothesis but also an alternative way to estimate a fluid's dynamic viscosity through a least-squares regression line. With future experimentation, these results can help prepare shock absorbers for vehicle and structural engineering practices.

INTRODUCTION

When one pours juice into a cup for breakfast, it flows quickly and freely out of the carton. However, when one pours maple syrup onto pancakes, it flows slowly and sticks to the pitcher. The contrast between these two liquids' behaviors arises from a difference in their viscosities. Viscosity is a quantity that expresses the magnitude of a fluid's internal friction or thickness (1). Every fluid possesses a dynamic viscosity, η, measured in units of pascal-seconds that depends inversely on its temperature and directly on the strength of its intermolecular forces (1). In the presented scenario, the juice has a low dynamic viscosity, while the maple syrup has a high dynamic viscosity. The motion of objects in the real world deals with the viscous properties of fluids, particularly in the case of damped simple harmonic motion. Examples of this phenomenon include the oscillations of a swing in the air and a piston in the oil of an automobile shock absorber.

Another well-studied mechanism that exhibits this type of motion is the vertical spring-mass system, which is a simple harmonic oscillator that consists of a spring of mass, m_{spring}, hanging from a ceiling with an attached sphere of radius, r, and mass, m_{sphere} (2). When a force is applied to the end of the spring, it either extends or compresses a distance known as the amplitude, A (2). Releasing this force causes the sphere to undergo undamped simple harmonic motion (SHM), which is when an object executes oscillatory motion subject to a restoring force proportional to its displacement but opposite in sign in the absence of an external force (2). Robert Hooke, in 1660, discovered a mathematical definition for this phenomenon known as Hooke's law (3). It states that:

$$F_R = -kx, \quad [1]$$

where F_R is the restoring force, k is a proportionality constant called the spring constant, and x is the object's displacement from equilibrium at $x = 0$ (3). In the case of the vertical spring-mass system, the spring exerts a restoring force on the sphere in the y-direction that causes it to perpetually complete revolutions about its equilibrium (2). Moreover, the spring's extension results in a negative displacement, its compression results in a positive displacement, and the absolute value of the sphere's maximum displacement is the amplitude (2).

When the sphere of the vertical spring-mass system executes SHM in a viscous fluid, it experiences a phenomenon known as damping. Damping is the exponential decay of a SHM system's mechanical energy due to external forces acting on the system (4). In this case, the fluid's viscous flow produces a drag force F_D on the sphere that gradually converts its mechanical energy into thermal energy (4). Unlike in a vacuum, where the sphere undergoes perpetual motion, damping effectively makes the sphere's oscillations dissipate over time (4). At slow speeds, this viscous drag force is approximately proportional to the sphere's velocity by the equation:

$$F_D \approx -b\frac{dy}{dt}, \quad [2]$$

where b is a proportionality constant called the damping coefficient (4).

The combination of this viscous drag force and the restoring force from equation [1] produces a differential equation that gives the solution for the sphere's position function, $y(t)$, in a viscous fluid as:

$$y(t) = A_0 e^{-\frac{b}{2m}t}\cos(\omega t + \phi) \quad [3]$$

at small values of b (**Figure 1**) (5). In this function, A_0 is the initial amplitude, m is the effective oscillating mass, b is the damping coefficient, ω is the angular frequency, and ϕ is the phase constant. The effective oscillating mass is the sum of

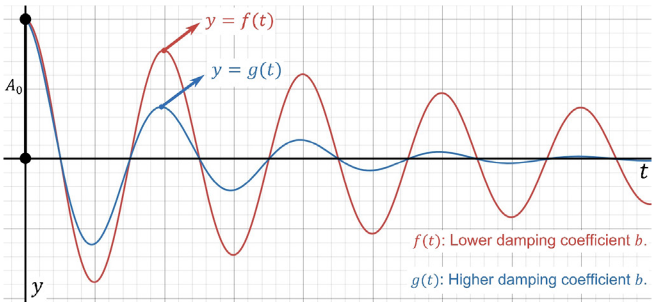

Figure 1: Damped simple harmonic motion illustration. The graph illustrates a red curve, $y = f(t)$, and a blue curve, $y = g(t)$, that depict the damped simple harmonic motion according to equation [3] of an object with a lower damping coefficient and one with a higher damping coefficient, respectively.

the sphere's mass and one-third of the spring's mass:

$$m = m_{\text{sphere}} + \frac{1}{3} m_{\text{spring}} \text{ (6).} \quad [4]$$

In addition, the angular frequency is a quantity that indicates the sphere's angular displacement per unit of time in which a full revolution is 2π radians (5). The phase constant is another quantity that expresses the amount of angular displacement of the sphere from equilibrium at $t = 0$ (5). Finally, the initial amplitude is the sphere's displacement from equilibrium at $t = 0$, assuming that $\phi = 0$, where successive amplitudes exponentially decay every new revolution (5).

Typically, an instrument known as the viscometer measures the dynamic viscosities of fluids by calculating either the terminal velocity of a falling sphere or the time taken for a liquid to flow through a capillary tube (7). However, a lesser-known viscometer involves determining the damping coefficient of a simple harmonic oscillator immersed in a fluid and employing an appropriate formula to measure the fluid's dynamic viscosity. Two researchers in fluid mechanics named Lev Landau and Evgeny Lifshitz developed theoretical work with Stokes' law to find such a formula (8). Stokes' law gives a fundamental relationship between the viscous drag force exerted on a sphere when falling at its terminal velocity, v_T, in a fluid and that fluid's dynamic viscosity as:

$$F_D = -6\pi r \eta v_T \text{ (9).} \quad [5]$$

Landau and Lifshitz expanded this definition to encompass the changing viscous drag force exerted on a sphere executing damped SHM at small amplitudes (8). Although many scientists have used their theoretical work to calculate various fluids' dynamic viscosities, it has often resulted in a sizable experimental error compared to their accepted values. For example, Peter Alexander and Evi Indelicato utilized a force sensor along with Landau and Lifshitz's theory to find the dynamic viscosity of water as 0.009 ± 0.004 Pa·s rather than 0.000915 ± 0.000010 Pa·s (6).

The inaccuracy in applying this theoretical work inspired us to create a more accurate empirical technique: a least-squares regression curve between the glycerol-water mixtures' dynamic viscosities and the corresponding damping coefficients of a simple harmonic oscillator immersed in those mixtures. One instance of research for this technique

Mixture number	Mass concentration, C_m (% by mass of glycerol)	Temperature, T (°C)	Dynamic viscosity η (Pa·s)	Damping coefficient b (N·s/m)
#1	0	21.2	9.76×10^{-4}	9.8141×10^{-3}
#2	10	22.8	1.21×10^{-3}	1.2212×10^{-2}
#3	20	22.8	1.61×10^{-3}	1.0942×10^{-2}
#4	30	22.8	2.26×10^{-3}	1.2508×10^{-2}
#5	40	22.8	3.36×10^{-3}	1.3513×10^{-2}
#6	50	23.5	5.28×10^{-3}	1.7546×10^{-2}
#7	60	22.9	9.64×10^{-3}	2.6076×10^{-2}
#8	70	22.9	2.00×10^{-2}	3.6842×10^{-2}
#9	80	22.8	5.15×10^{-2}	5.8461×10^{-2}
#10	90	24.2	1.65×10^{-1}	1.1853×10^{-1}
#11	100	22.8	1.10	4.7133×10^{-1}

Table 1: Mass concentrations, temperatures, dynamic viscosities, and damping coefficients of glycerol-water mixtures.

was by Milad Radiom, who found a direct linear relationship but did not provide an equation for the regression and had a very small sample size of $n = 4$ (10). This direct relationship arises from how a higher dynamic viscosity creates more flow resistance exerted on the simple harmonic oscillator, causing the mechanical energy of its oscillations to dissipate more quickly and resulting in a higher damping coefficient. Given this, we hypothesized that the relationship between the damping coefficient of a simple harmonic oscillator and a fluid's dynamic viscosity was positively correlated. We tested this hypothesis by constructing a vertical spring-mass system, preparing eleven glycerol-water mixtures of varying mass concentrations, filming the damped SHM of the oscillator in these mixtures, and correlating the mixtures' dynamic viscosities and the oscillator's damping coefficients with statistical inference. Our experiments not only verified our above claim but also provided a least-squares regression line equation. With future experimentation and a collection of more data points, this equation may be practical in determining a fluid's dynamic viscosity for a given damping coefficient, which has applications to shock absorbers in both vehicle and structural engineering.

RESULTS

Glycerol is a clear, colorless, and viscous liquid that belongs to the alcohol family of organic compounds (11). For

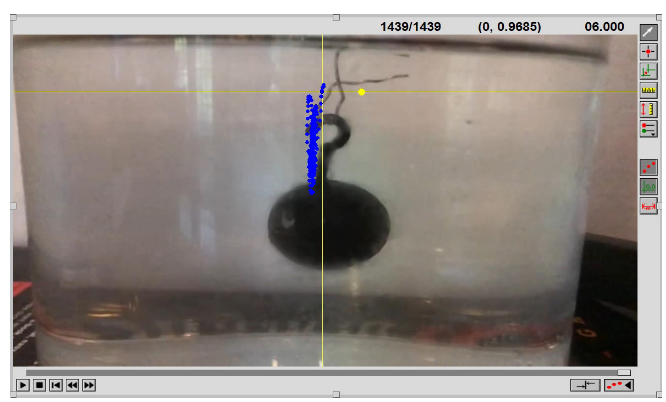

Figure 2: Vernier's Logger Pro video analysis feature. The blue dots represent the collection of points (x, y) for the vertical spring-mass system's sphere in a glycerol-water mixture (#9 in this case) with respect to the origin, which is the point of intersection of the two yellow lines. Each point was collected every five frames until the number of frames given by the number on the far left of the top toolbar (1439 in this case) was reached.

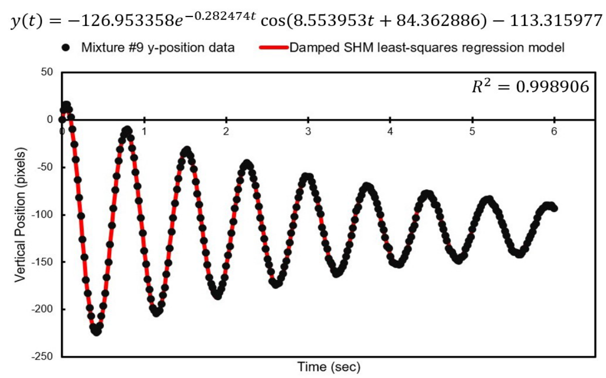

Figure 3: Least-squares regression on *y*-position data. The graph depicts the sphere's *y*-position at different points in time (*t, y*) as black dots and the least-squares regression curve to model that data as a red curve for a glycerol-water mixture (#9 in this case). The data was plotted through the collection of points in Logger Pro's video analysis feature, and the regression was performed through Logger Pro's data analysis feature to six decimal places for the equation and the correlation coefficient of determination displayed on the top.

the first step, this liquid was mixed with water to create eleven glycerol-water mixtures of varying dynamic viscosities, where glycerol was the solute and water was the solvent. These dynamic viscosities were calculated based on their dependencies with the mixtures' mass concentrations and temperatures expressed in equations [6] through [11]. For example, a glycerol-water mixture with a 10% mass concentration and a 22.8°C temperature had a dynamic viscosity of 1.21 x 10^{-3} Pa·s (**Table 1**).

For the second step, the sphere of a self-constructed vertical spring-mass system with an effective oscillating mass of 0.10348 kg was filmed executing damped SHM in each glycerol-water mixture at 240 frames per second for 6 seconds. These videos were inserted into Vernier's Logger Pro through its video analysis feature (**Figure 2, Figure S1**), and and data points were collected for the sphere's *y*-position (**Figure 3, Figure S2**) and *x*-position (**Figure 4, Figure S3**) every five frames. A least-squares regression according to equation [12] was then utilized on the *y*-position data of each video to form the predicted *y*-position as a function of time, $\widehat{y(t)}$. These least-squares regressions were all highly correlated ($R^2 \geq 0.997623$), and every parameter was calculated to six decimal places (**Figure 3, Figure S2**). From each least-squares regression, the damping coefficient was calculated based on equation [13]. For instance, the damping coefficient of the vertical spring-mass system in the second glycerol-water mixture was 1.2212 x 10^{-2} N·s/m (**Table 1**).

For the final step, the dynamic viscosities for the glycerol-water mixtures and the corresponding damping coefficients for the vertical spring-mass system were plotted as points (η,b). A least-squares linear regression line was applied to these points that resulted in a very high positive correlation (r = 0.995) between the two variables. The regression summarized that as the dynamic viscosity increased by 1 Pa·s, the damping coefficient of the vertical spring-mass system increased by 0.415 N·s/m, on average (**Figure 5**). However, this linear regression was not the most appropriate curve fit because the data points for the low dynamic viscosities behaved quite differently from the data points for the higher dynamic viscosities. Specifically, the data points with low dynamic viscosities appeared to increase at a faster rate than the data points with high dynamic viscosities, and

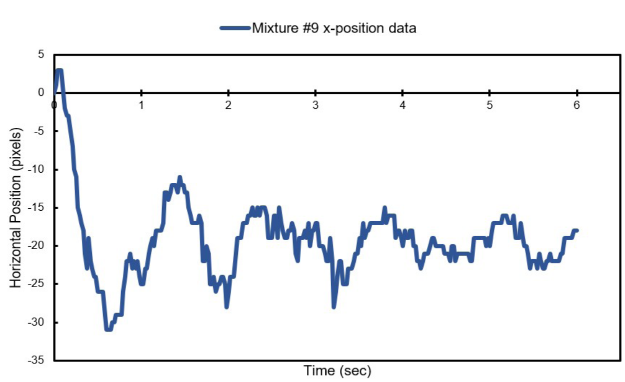

Figure 4: Data for *x*-position. The graph depicts the sphere's *x*-position at different points in time (*t, x*) as a blue line chart for a glycerol-water mixture (#9 in this case). The data was plotted through the collection of points in Logger Pro's video analysis feature.

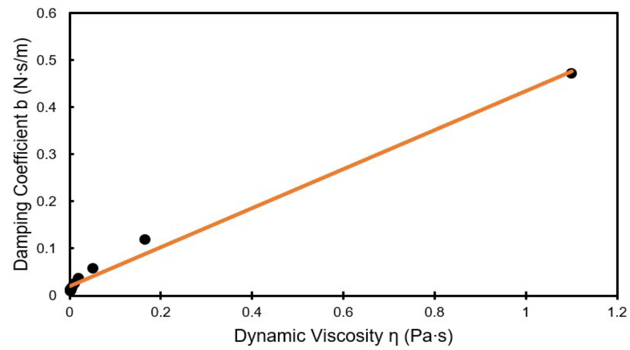

Figure 5: Scatterplot of dynamic viscosity and damping coefficient. The graph illustrates a scatterplot of the black points (η,b) depicting the relationship between the glycerol-water mixtures' dynamic viscosities and their corresponding damping coefficients of the vertical spring-mass system. A linear least-squares orange regression curve models these points in the equation $\hat{b}$=0.415η+0.0203 with a positive correlation of r = 0.995.

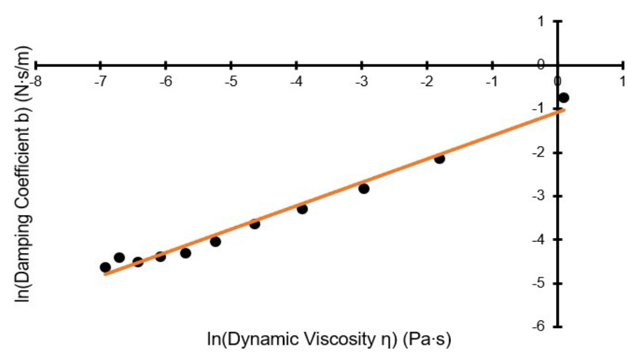

Figure 6: Double-log scatterplot of dynamic viscosity and damping coefficient. The graph illustrates a double-logarithm scatterplot of the black points (ln(η), ln(b)) along with a linear least-squares orange regression curve that models these points in the equation $\widehat{\ln(b)}$ =0.537ln(η)-1.08 with a positive correlation of r=0.990. This equation can also be written in the form of the power curve $\hat{b}$=0.338$\eta^{0.537}$. This positive linear correlation was determined to be statistically significant through a t-test for slope (p = 2.87 x 10^{-9}).

since these clusters of points are so far away from each other, their disagreement could have created bias in the correlation coefficient. Therefore, a double-logarithm regression was incorporated instead to account for this decreasing rate of change, which also made it easier to see the finer differences between the data points with low dynamic viscosities. The double-logarithm regression summarized that a very high positive correlation (r = 0.990) existed between the natural logarithm of these two variables, whereby an increase of 1 Pa·s of the natural logarithm of the dynamic viscosity resulted in an increase of 0.537 N·s/m of the natural logarithm of the damping coefficient, on average (**Figure 6**).

To verify that this positive correlation did not occur by chance, a one-tailed t-test for slope was conducted on the sample regression. Using the LinRegTTest method on a TI-84, the positive correlation was determined to be statistically significant at an alpha-level of 0.05, which suggests that there exists a positive correlation between the natural logarithm of a damping coefficient of a simple harmonic oscillator and the natural logarithm of a fluid's dynamic viscosity for a population of glycerol-water mixtures ($p = 2.87 \times 10^{-9}$).

DISCUSSION

The results from this experiment do not necessarily support the original hypothesis of a direct relationship but rather a modified hypothesis: there exists a positive linear relationship between the natural logarithm of a fluid's dynamic viscosity and the natural logarithm of a damping coefficient of a simple harmonic oscillator. In other words, the relationship between the two variables is not linear in nature but rather based on a power regression. The sample regression of this double-logarithm scatterplot of eleven data points produced a positive slope value of 0.537, which was determined to be statistically significant. The p-value was 2.87×10^{-9}, which indicates that there is a 2.87×10^{-7} percent chance that results as extreme as this would be observed again assuming that the slope between these two variables is zero was true. Since this probability is well below the alpha-level threshold of 0.05, the proposed hypothesis of a positive slope between these two variables is more likely to be true.

Although the proposed hypothesis was successfully confirmed, possible sources of error still exist that could have influenced the results. One such error is the sphere's horizontal motion in the x-direction (**Figure 4, Figure S3**), which could have influenced the values of the damping coefficients (**Table 1**). This horizontal motion indicates that the viscous drag force from equation [2] also acts in the x-direction, meaning the sphere's motion from the vertical spring-mass system resembles a pendulum or a parametric oscillator. Consequently, equation [3] and the damping coefficients are only approximate because the equation was derived assuming that the viscous drag force only acts in the y-direction. However, the excellent least-squares regression curve fit implies that this source of error was trivial. A final example of another source of error is the imprecise nature of Vernier's Logger Pro video analysis feature. The points (x,y) must be collected by the user clicking at the same point of the sphere every five frames with a cursor (**Figure 2, Figure S1**). Since this method involves a qualitative observation of the sphere's location, some margin of error likely existed between the sphere's true position and the observed position by clicking.

Nonetheless, despite these sources of error, these results have some significant implications for engineering and future scientific work. Traditionally, a liquid's dynamic viscosity is calculated through a few experimental methods, such as measuring either the terminal velocity of a falling sphere or the time taken for a liquid to flow through a capillary tube (7). This method can be added to the list of these existing techniques. Due to the statistical significance and high correlation coefficient of the sample double-logarithm regression line, the explanatory variable η can act as an effective predictor for the response variable b (**Figure 6**). In other words, one could calculate the predicted damping coefficient of a simple harmonic oscillator given a fluid's dynamic viscosity or vice versa.

This least-squares regression line equation from the double-logarithm scatterplot has many potential applications in preparing shock absorbers for engineering practices. Shock absorbers are mechanical devices that dampen an object's oscillations caused by a collision by dissipating its kinetic energy (12). The most common shock absorber is the viscous damper, which consists of a piston inside a pressurized cylinder with a viscous fluid that travels between two internal chambers, converting an object's oscillatory mechanical energy into thermal energy dissipated into the atmosphere (12). Vehicle engineers develop these shock absorbers with springs coiled around them attached to a car's suspension system to improve a ride's smoothness (12). Specifically, when a vehicle drives over a bump in the road, the spring compresses with elastic potential energy and then rebounds with kinetic energy that dissipates into thermal energy over time (12). Without the shock absorber, the spring and the vehicle's suspension system would continuously oscillate in undamped simple harmonic motion, causing a very bouncy car ride (12). Furthermore, structural engineers utilize these shock absorbers in the foundations of buildings to mitigate the effect of earthquakes and strong wind that causes buildings to shake and ultimately collapse (12). These engineers need to know the damping coefficients of their shock absorbers to quantify the effect of damping on their systems. They often design shock absorbers to be in a state called critical damping, whereby a system returns to its equilibrium position after a collision in the fastest way possible without overshooting into oscillation (13). Mathematically, this condition occurs when the square of a system's damping coefficient is equal to four times the product of its mass and spring constant (5). Although engineers usually manipulate the spring constant to yield this critical damping coefficient, adjusting the value of the fluid's dynamic viscosity of a shock absorber based on the least-squares regression line equation offers an alternative way to produce the same result. In essence, this least-squares regression line equation could allow engineers to set the damping coefficient of a shock absorber by calculating its fluid's dynamic viscosity. However, further potential experiments would be necessary to maximize this calculation's accuracy, such as finding a way to reduce the sphere's horizontal motion, discovering a technology to calculate the position of the sphere's center of mass every frame, or increasing the sample size of the number of glycerol-water mixtures. It is also worth conducting the measurement at a particular dynamic viscosity several times, as with more data, the error associated with these issues can be better quantified as error bars.

Overall, the results from this experiment suggested that a fluid's dynamic viscosity and the damping coefficient of an object's oscillations are positively correlated. These results were found through filming the damped SHM of a vertical spring-mass system's sphere in various glycerol-water mixtures and analyzing those videos in Vernier's Logger Pro. Although there were some possible sources of error, the results demonstrate that using this experiment can predict a fluid's dynamic viscosity based on a determined damping coefficient, which has applications to structural and vehicle engineering.

MATERIALS AND METHODS

Construction of the Vertical Spring-Mass System

A 24 in length and 12 mm diameter steel rod fastened to a 9 in by 9 in powder coated pressed steel triangular base (EISCO, SKU TRISTD2) was mounted onto a countertop. A pendulum clamp of a 11.5 in length (EISCO, SKU PH0308) was then attached to the steel rod that suspended a low-friction harmonic spring with a 7 in length, a 1.4 in diameter, a spring constant of 9.6 N/m, and a mass of 105.20 g (EISCO, SKU PH0713). Finally, 22-gauge steel wire was connected to an iron sphere with a 1 in diameter and a mass of 68.41 g (EISCO, SKU PH0306F) and to the end of the spring. This steel wire was necessary to ensure that the spring did not submerge itself in the fluid.

Preparation of the Fluids

The sphere of the vertical spring-mass system was immersed in eleven different glycerol-water mixtures of varying mass concentrations, C_m, contained in a glass jar, where the glycerol was the solute and the water was the solvent. Each mixture had a total mass of 1700 g and a mass concentration of glycerol between 0% and 100% with increments of 10%. The glycerol came from a vegetable glycerol bulk gallon container of 99.7% glycerol content and 0.3% water content (Raw Plus Rare, ASIN B06XXHXRND). These mixtures had dynamic viscosities dependent on their mass concentrations and their temperatures, *T*, according to empirical formulas developed by Nian-Sheng Cheng of Zhejiang University (14). These formulas are as follows:

$$A = 0.705 - 0.0017T; \quad [6]$$

$$B = (4.9 + 0.036T)A^{2.5}; \quad [7]$$

$$\alpha = 1 - C_m + \frac{ABC_m(1 - C_m)}{AC_m + B(1 - C_m)}; \quad [8]$$

$$\eta_{\text{water}} = 1.790e^{\left(\frac{(-1230-T)T}{36100+360T}\right)}; \quad [9]$$

$$\eta_{\text{glycerol}} = 12100e^{\left(\frac{(-1233+T)T}{9900+70T}\right)}; \quad [10]$$

$$\eta = \eta_{\text{water}}^{\alpha}\eta_{\text{glycerol}}^{1-\alpha}. \quad [11]$$

Using these formulas, a function was coded in Python to return the dynamic viscosity of a glycerol-water mixture given the parameters of its mass concentration and its temperature to simplify the calculation process (**Appendix B**).

Filming of the Damped Simple Harmonic Motion

The sphere of the vertical spring-mass system executing damped SHM in each of these glycerol-water mixtures was filmed at 240 frames per second with the slow-motion feature of an iPhone camera. After the filming, the videos were trimmed at 6 seconds (4 seconds for the 90% mass concentration and 1.5 seconds for the 100% mass concentration), and the videos' frames were cropped to observe only the region of interest. Furthermore, the sphere was painted black and was filmed oscillating against a white cardboard background to reduce light reflections and noise levels. Next, the videos were inserted into Vernier's Logger Pro software through the movie analysis feature, and the points (*x, y*) were collected at the same spot of the sphere every five successive frames of its oscillations. Logger Pro then plotted the points (*t, x*) and (*t, y*) on two separate graphs, allowing for a least-squares regression to be employed to model the sphere's *y*-position as a function of time using a variation of equation [3]. The regression utilized was:

$$\widehat{y(t)} = A_0e^{-\gamma t}\cos(\omega t + \phi) + K, \quad [12]$$

where *K* is simply a vertical shift constant in the position function's graph. The damping coefficient, *b*, can be calculated from *γ* by the equation:

$$\gamma = \frac{b}{2m}. \quad [13]$$

Correlation Between the Dynamic Viscosity and the Damping Coefficient

Finally, once the eleven values of the glycerol-water mixtures' dynamic viscosities and the vertical spring-mass system's damping coefficients were calculated, the points (η,b) and (ln(η),ln(b)) were plotted. Least-squares linear regressions were then utilized on both scatterplots to calculate the regression equations and the correlation coefficients, *r*. The double-log scatterplot was ultimately selected due to the decreasing rate of change nature of the original scatterplot. Based on the regression line's slope, the residual standard error, the standard deviation of the natural logarithm of the dynamic viscosities, and the sample size, a *t*-test for slope was used to determine the statistical significance of the correlation in the double-log scatterplot. This *t*-test was conducted using the LinRegTTest function on the TI-84 calculator, which calculated the *p*-value of the proposed hypothesis that ln(*b*) and ln(η) were positively correlated against a hypothesis of no correlation. In other words, it tested for $\beta > 0$ against $\beta = 0$, where β is the slope of a population regression line between these two variables. This *p*-value was ultimately used to confirm or reject the proposed null hypothesis.

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Appendix A – Supplemental Figures

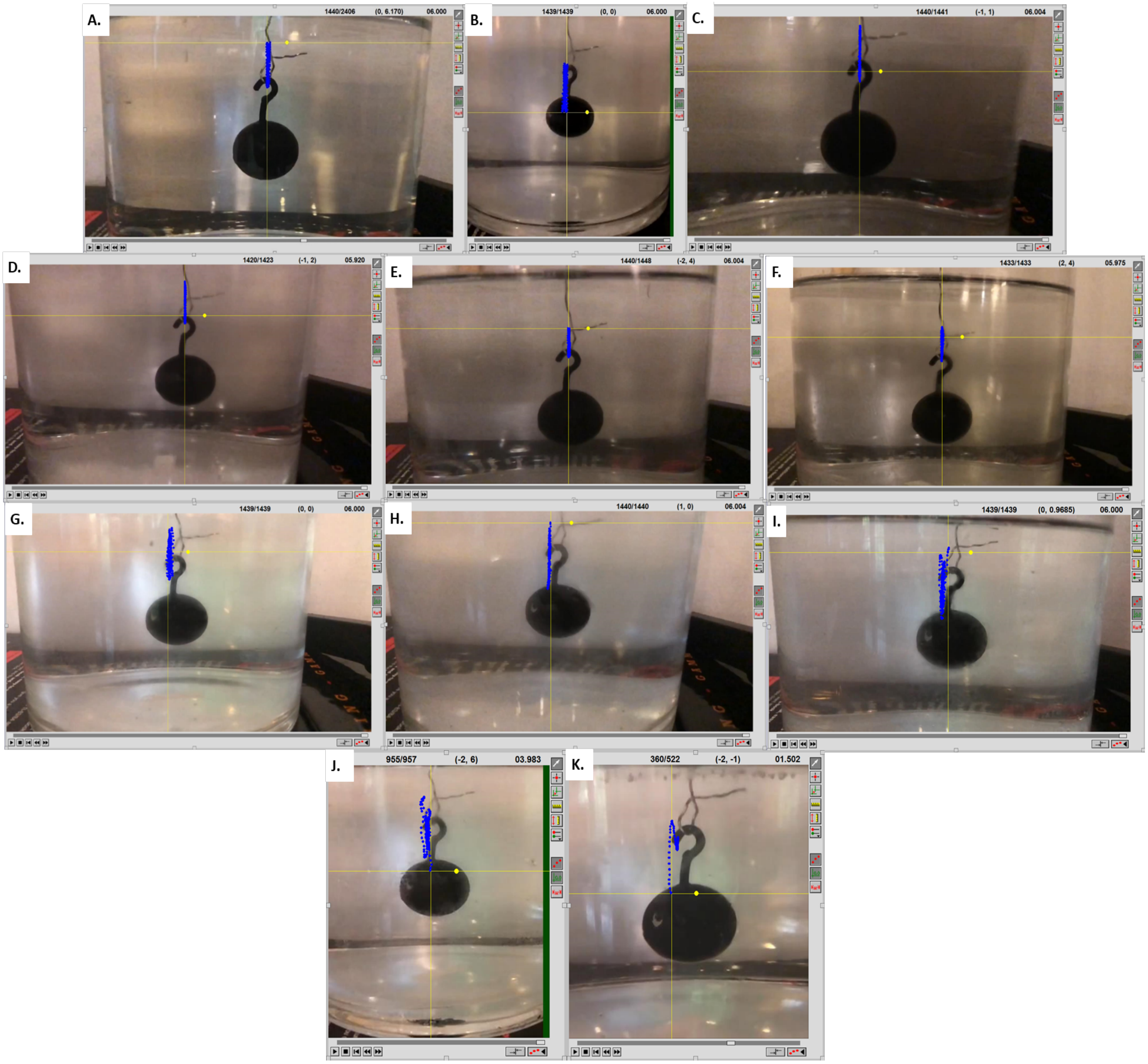

Figure S1. Vernier's Logger Pro video analysis feature. A) Mixture #1, **B)** Mixture #2, **C)** Mixture #3, **D)** Mixture #4, **E)** Mixture #5, **F)** Mixture #6, **G)** Mixture #7, **H)** Mixture #8, **I)** Mixture #9, **J)** Mixture #10, **K)** Mixture #11. Blue dots represent the collection of points (*x, y*) for the vertical spring-mass system's sphere in the eleven glycerol-water mixtures with respect to the origin, which is the point of intersection of the two yellow lines. Each point was collected at every fifth successive frame until the number of frames given by the number on the far left of the top toolbar was reached.

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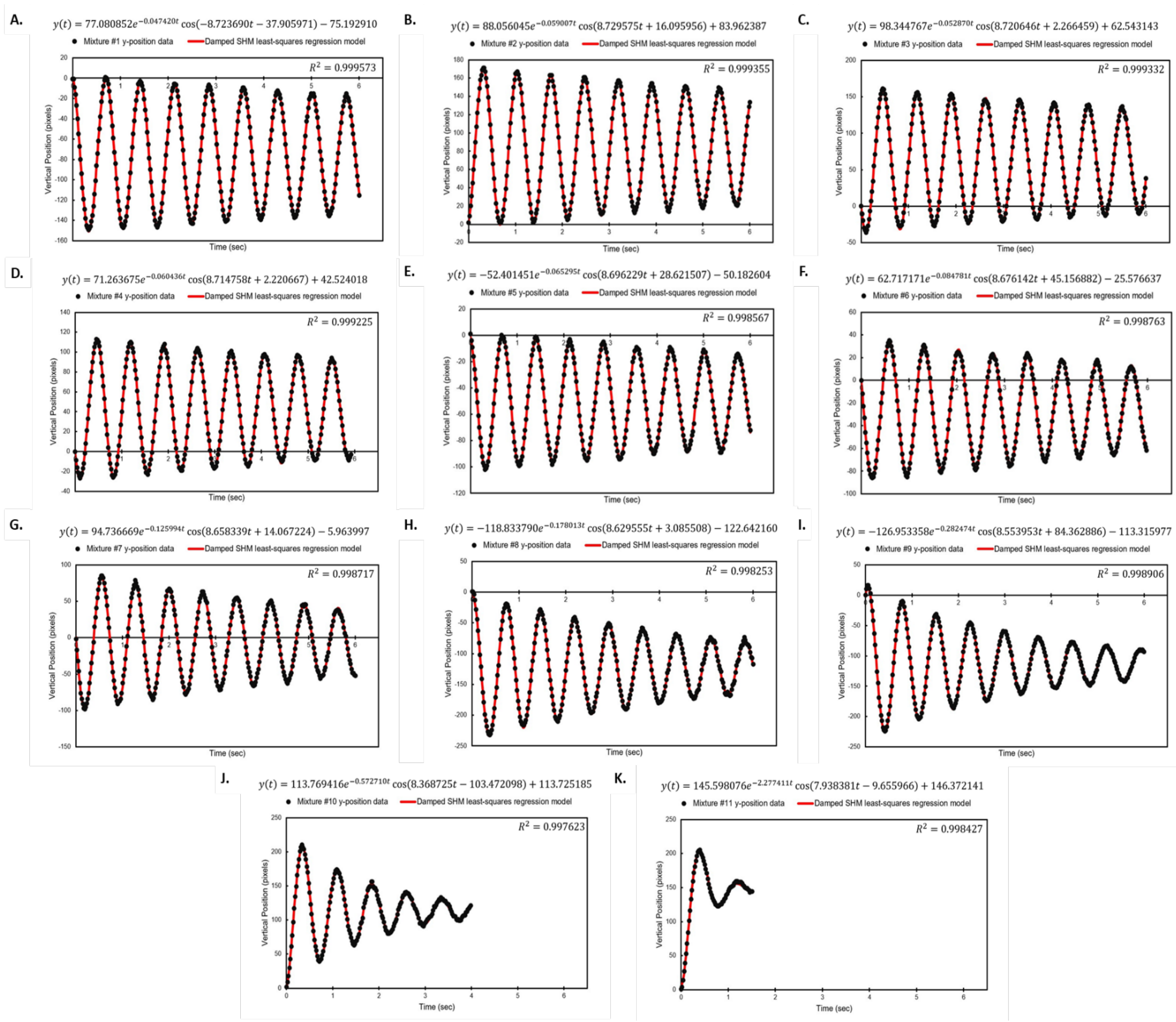

Figure S2. Least-squares regression on *y*-position data. A) Mixture #1, **B)** Mixture #2, **C)** Mixture #3, **D)** Mixture #4, **E)** Mixture #5, **F)** Mixture #6, **G)** Mixture #7, **H)** Mixture #8, **I)** Mixture #9, **J)** Mixture #10, **K)** Mixture #11. Graphs depict the sphere's *y*-position at different points in time (*t, y*) as black dots and the least-squares regression curve to model that data as a red curve for the eleven glycerol-water mixtures. The data was plotted through the collection of points in Logger Pro's video analysis feature, and the regression was performed through Logger Pro's data analysis feature to six decimal places for the equation and the correlation coefficient of determination displayed on the top.

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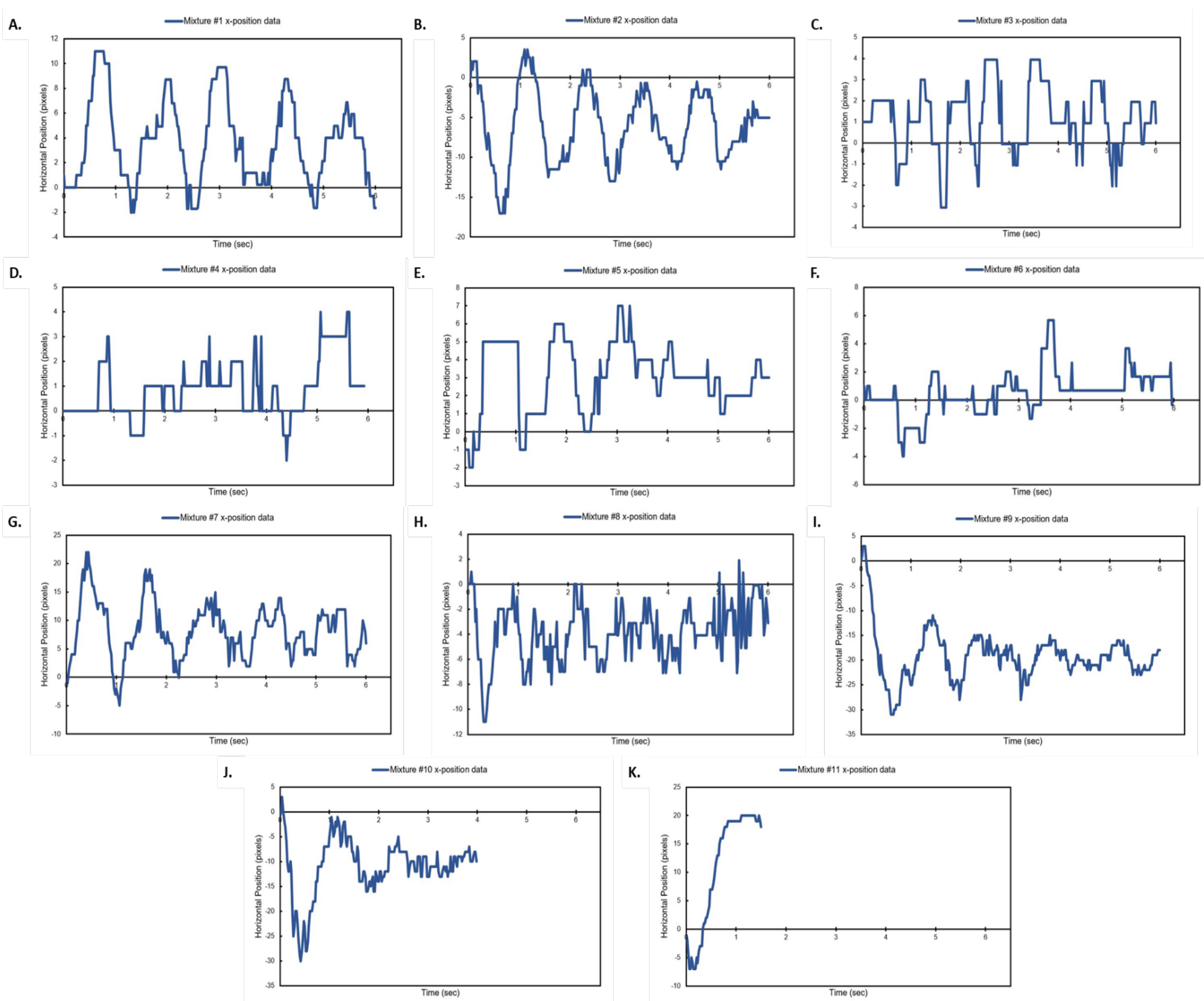

Figure S3. Data for *x*-position. A) Mixture #1, **B)** Mixture #2, **C)** Mixture #3, **D)** Mixture #4, **E)** Mixture #5, **F)** Mixture #6, **G)** Mixture #7, **H)** Mixture #8, **I)** Mixture #9, **J)** Mixture #10, **K)** Mixture #11. The graph depicts the sphere's *x*-position at different points in time (*t, x*) as a blue line chart for the eleven glycerol-water mixtures. The data was plotted through the collection of points in Logger Pro's video analysis feature.

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Appendix B – Python Code

The Python code shows a function that returns the dynamic viscosity of a glycerol-water mixture given the parameters of its mass concentration and temperature using equations [6] through [11].

```python
'''
Authors:
- Richard Michael Powell III
- Richard Michael Powell Jr.

Narrative:
- This program allows the user to calculate the dynamic viscosity of a glycerol-water mixture given its mass concentration and temperature.
'''

# Imported packages
import math

def dynamic_viscosity_calculator(T, C):
    '''
        Summary and description of function:
        - This function returns the dynamic viscosity of a glycerol-water mixture given the mixture's mass concentration and temperature.

        Parameters:
        - T: The temperature of the mixture in degrees Celsius (float)
        - C: The mass concentration of the mixture as a decimal (float)

        Returns:
        - dynamic_viscosity: The dynamic viscosity of the mixture in pascal-seconds (float)
    '''

    # Calculations for the additional parameters
    a = 0.705-0.0017*T
    b = (4.9+0.036*T)*math.pow(a, 2.5)
    alpha = 1-C+((a*b*C*(1-C))/(a*C+b*(1-C)))

    # Calculations for the dynamic viscosities of pure substances
    viscosity_water = 1.790*math.exp(((-1230-T)*T)/(36100+360*T))
    viscosity_glycerol = 12100*math.exp(((-1233+T)*T)/(9900+70*T))

    # Calculation for the dynamic viscosity of the glycerol-water mixture
    dynamic_viscosity = (math.pow(viscosity_water, alpha)*math.pow(viscosity_glycerol, 1-alpha))/(1000)
    return dynamic_viscosity


def main():
    # Inputs to collect data on the temperature and mass concentration
    T = float(input('Insert the temperature of the glycerol-water mixture.\n'))
    C = float(input('Insert the mass concentration of the glycerol-water mixture.\n'))

    # Output for the dynamic viscosity
    dynamic_viscosity = dynamic_viscosity_calculator(T, C)
    print('The dynamic viscosity of the glycerol-water mixture is ' + str(dynamic_viscosity) + ' Pa*s.')

if __name__ == '__main__':
    main()
```