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Article

Heat conduction: Mathematical modeling and experimental data

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SUMMARY

Mathematical models have been used to study and understand many biological and physical systems. However, for mathematical models to be useful, one needs to carefully assess the assumptions that go into the derivation of the mathematical model. For example, the heat equation, which is fundamental to the study of heat transfer, was derived based on an empirical rate formulation known as Fourier's law of conduction and perfect insulation. Since perfect insulation is very difficult to implement in laboratory conditions, we hypothesized that the heat equation would produce a time course temperature solution that overestimates real temperature data as a consequence of heat loss to the surrounding air, which is not accounted for in the heat equation. To test this hypothesis, we derived two mathematical models for the rate of change of heat under two laboratory conditions: one that considers heat loss and one that does not. The experiment consists of a squared metal bar that was heated at one end by a soldering iron and the temperatures along its length were measured by thermocouples. Our results, when compared to experimental data, clearly demonstrated that the mathematical model obtained by taking into consideration the heat loss yielded a better fit to the experimental data than the model without the heat loss assumption. Specifically, the mathematical model with the assumption of perfect insulation overestimated the temperature data. The research findings showed that when using a mathematical model, it is important to examine carefully the assumptions made in deriving the mathematical model.

INTRODUCTION

Heat transfer is one of the most common physical phenomena in our daily lives, including heating and cooling systems, engine radiators in cars, and heat pipes in computer systems (1). In general, heat transfer is energy in transit due to temperature differences, and hence "energy balance" is the underlying conservation principle (1, 2). The rate of heat flow, or rate of heat transfer, is the amount of heat transferred per unit of time in some particular material, usually measured in watts (joules per second). One of the most important factors affecting the rate of transfer, other than the temperature difference between two locations, is the material involved (2, 3). This dependence of heat transfer on the material property is often expressed in terms of a parameter known as the thermal conductivity (3).

Significant technological advancements have occurred in the past 20 years, making investigations of heat transfer rates regarding different materials rich in societal applications. For instance, in the United States, the generation of electricity mainly comes from three sources: fossil fuels, nuclear energy, and renewable energy (4). Heat generated in a nuclear reactor is transferred to water surrounding the reactor. The water then, in the form of steam, carries the heat to a steam turbine where electricity is finally produced. The key point is to convey the heat from the reactor to the water with as little heat loss as possible, and this requires using materials with the highest heat conductivity to increase the heat transfer rate (5).

Fundamental to the study of heat transfer is the heat equation, which was formulated at the beginning of the 19th century by French mathematician and physicist Joseph Fourier. Fourier's work has inspired other researchers to use the heat equation to solve a variety of problems in probability, financial mathematics, and quantum physics, as well as problems in biological and social sciences (6). The heat equation was derived based on an empirical rate formulation known as Fourier's law of conduction and perfect insulation, which assumes no heat escapes the solid as it travels from hot to cold regions (2). The heat equation has the following form:

$$\frac{d}{dt}u(t,x) = \left(\frac{K}{\rho c_p}\right)\cdot\frac{d^2}{dx^2}\,u(t,x) \qquad \textbf{(1)}$$

Here, the variable represents the temperature at time *t* and location *x*, and the coefficient $\alpha = K/(\rho c_p)$is known as thermal diffusivity in units of m^2/sec, which represents the ability of a material to conduct heat relative to its ability to store heat. In practice, however, perfect insulation is very difficult to attain in experimental laboratory conditions. We hypothesized that the heat equation produces a time course temperature solution that overestimates real temperature data since it does not take into account heat loss to the surrounding air. To test this hypothesis, we used another empirical observation, Newton's law of cooling, to derive a modified heat equation given by the following expression:

$$\frac{d}{dt}\left(c_p\rho\, u(t,x)\right) = K\cdot\frac{d^2}{dx^2}u(t,x) - \frac{2h(a+b)}{ab}\left(u(t,x) - u_{air}\right) \textbf{(2)}$$

The additional term on the right-hand side of Equation 2 accounts for heat loss due to convection derived from Newton's law of cooling. The derivation of the heat and modified heat equations is presented in the Appendix section. We tested both models against experimental data collected from a heat experiment. The heat experiment consists of a square metal bar that was heated by a soldering iron encapsulated cylindrical heater and the temperatures along the length of the bar were measured by thermocouples mounted at multiple locations on the bar. Using MATLAB, we demonstrated that the modified heat equation provided a better fit to the experimental data than the original heat equation, while the original heat equation overestimated the real temperature data since it did not account for the heat loss to the surrounding air. The research findings showed the importance of examining carefully the assumptions went into the derivation of the mathematical model. If the application does not satisfy those assumptions, one cannot rely on the mathematical model to give a reliable prediction on the application phenomena.

RESULTS

In this section, we present simulation results of the steady-state temperature solution from the heat equation A23 and the modified heat equation A24 obtained from the software package MATLAB (MathWorks). All physical parameters in the models – the thermal conductivity K (W/(mK)), the heat flux f (W/m^2), and the Newton cooling constant h (W/(m^{2}K)) – were estimated from experimental data using a least-squares curve fitting formulation. The curve fitting procedure and the experimental set-up to obtain the data are described in the Methods section. To make sure that the temperature data reach steady-state values, the experiment was running sufficiently long enough so that 10 temperature data obtained at 0.1 sec. intervals were relatively constant from each thermocouple.

Original Heat Equation

We begin by determining how well the steady-state temperature of the copper rod, which was obtained from the solution of the original heat equation A23, fits the experimental data. The model solution using estimated parameters K = 410.5 W/(mK) and f = -26550 W/m^2 were plotted against the steady-state temperature data (**Figure 1**). Here, the parameters K and f represent thermal conductivity and heat flux, respectively, and were obtained by solving a least-squared curve fitting problem using MATLAB function ***fminsearch.*** The temperature from the original heat equation very poorly fitted the experimental data. The model underestimated the data at the beginning of the rod where the heat source is located and overestimated the data elsewhere in the rod, clearly demonstrating that the perfectly insulated assumption on the rod used in the derivation of the copper rod is not appropriate, as the model produced higher temperatures than experimental data in the region outside of the heat source. The experimental data (**Figure 1**) showed that there are heat losses as the heat is transferred along the length of the rod. In addition, the limitation of the linear steady-state solution deriving from the original heat equation, the specified boundary conditions, and the unique characteristics of the least-squares curve-fitting algorithm contributes to both the over- and underestimation.

Modified Heat Equation

Next, we examined how well the steady-state temperature derived from the modified heat equation (taking heat loss into consideration), equation A24, fit the same experimental data for the copper rod. The modified heat equation, with estimated parameters K = 408.28 W/(mK), f = -150000 W/m^2, and h = 13.6 W/(m^{2}K), produced an almost exact fit to the experimental data (**Figure 2**). Here, the parameter h denotes the Newton cooling constant.

Finally, we presented a comparison of the modified heat equation model to the experimental data for the aluminum rod. For comparison, we provided the results between the modified heat equation, with estimated parameters K = 195 W/(mK), f = -206000 W/m^2, and h = 18.3 W/(m^{2}K), and the experimental data for aluminum (**Figure 3**). We have noted that the fit is excellent, similar to the case of the copper rod; however, the temperature for the aluminum rod compared to the copper rod is higher at the source and lower outside

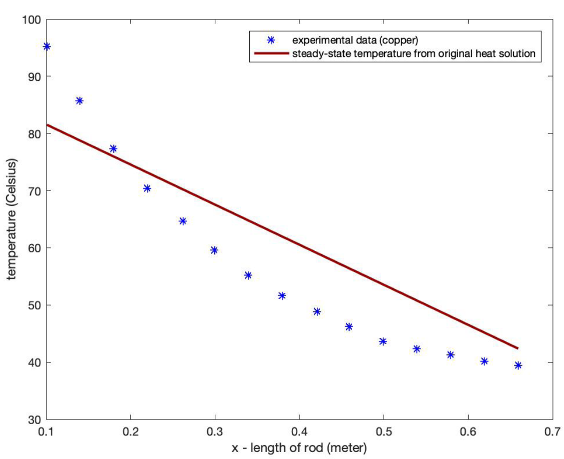

Figure 1: Comparison of copper experimental data and fit to the original heat equation. The curve depicted by the original heat equation clearly had a relatively great deviation with the experimental data points, as it failed to take into account some real-world factors. These unconsidered factors, including heat loss to the surrounding air, exert an effect most prominently seeing at the beginning and the middle parts of the graph.

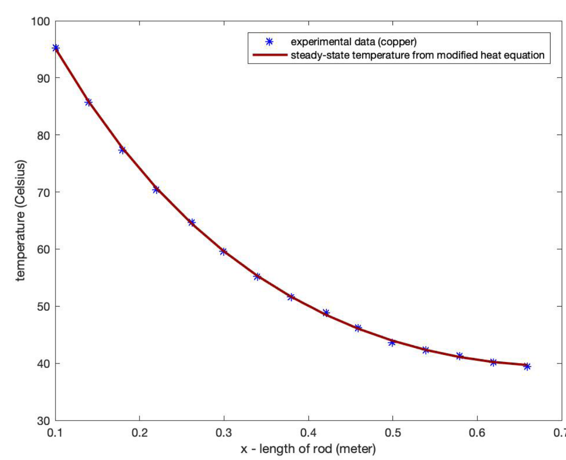

Figure 2: Comparison of copper experimental data and fit to the modified heat equation. The modified heat equation evidently produced a better estimation of the real-world conditions, as the curve portraited by the equation nearly overlapped with each data point we obtained from our experiment.

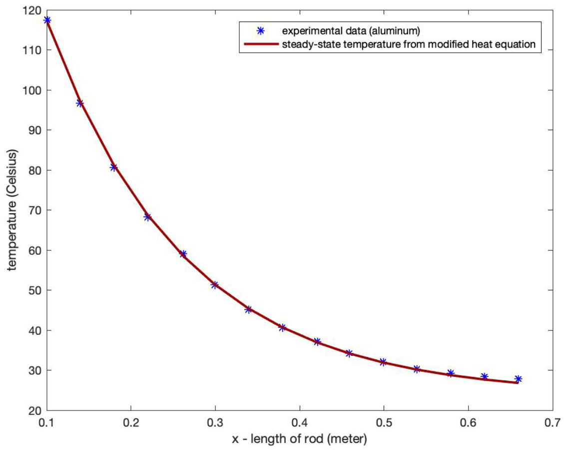

Figure 3: Comparison of aluminum experimental data and fit to the modified heat equation. The modified heat equation produced a nearly perfit fit to all the experimental data for the aluminum bar. It is noted that the temperatures toward the right end of the rod, near x = 0.66 m, are lower than those recorded for the copper rod. This is a consequence of the fact that aluminum has a smaller thermal conductivity value than copper.

of the source. The research findings demonstrated that the aluminum rod is not as efficient as copper at conducting heat, which is confirmed since we obtained a lower thermal conductivity for the aluminum than the copper rod (K = 195 W/(mK) for aluminum versus K = 410.5 W/(mK) for the copper rod). We noted that our estimated thermal conductivity value for aluminum 195 W/(mK) agrees well with the literature value of 237 W/(mK) (7).

DISCUSSION

Mathematical models, in general, contain parameters with unknown values. For the original and modified heat equations, the unknown parameters are the thermal conductivity K (W/(mK)), the heat flux f (W/m^2), and the Newton cooling constant h (W/(m^{2}K)). A common approach to determine the parameter values from the experimental data is by curve fitting. However, depending on the structure of the mathematical model, the parameter values obtained by curve fitting might not be unique even though the mathematical model solution and the experimental data fit nicely.

Original Heat Equation

If the parameter values cannot be obtained uniquely, different initial guesses of the parameter values would produce different parameter values, including those values that are not physical. For example, using initial values for K = 10 W/(mK) and f = -50 W/m^2, the estimated values of K and f that were determined by MATLAB routine ***fminsearch*** were 1.15 W/(mK) and -74.38 W/m^2, respectively. The estimated thermal conductivity for copper was 1.15 W/(mK), which is far off from the literature value of 380 W/(mK) for commercial copper at 363 K (8). It is noted that K and f appear as a quotient in the temperature expression, Equation A23 derived from the original heat equation. Its estimated quotient value f/K was -64.68. Since K and f appear as a quotient in the Equation A23, their estimated values are not unique. For example, when we picked the following initial guesses K = 250 W/(mK) and f = 100000 W/m^2, the estimated parameters K and f that minimized the residual errors, Equation 3, by ***fminsearch*** were 410.5 and –26550, respectively. Their quotient value is -64.68, which is the same as for the other initial guesses, but the value for the thermal conductivity 410.5 W/(mK) is much closer to the literature value 380 W/(mK).

We note that even with a thermal conductivity value that is close to the literature value, the fit of the steady-state temperature comparing to the experimental data is not good. The model mostly overestimated the experimental data (**Figure 1**). This firmly supports our hypothesis that the original heat equation produced a time course temperature that mostly overestimated the real temperature data since it does not consider the heat loss to the surrounding air. To determine a better estimation of the heat flux at the left end point of the bar, we assume the literature value 380 W/(mK) for the thermal conductivity of the copper bar and estimate only the heat flux f. This yields a heat flux f value to be -24579 W/m^2. We note that the fit quotient f/K still comes out to be -64.68, which is the same as the other estimates.

Modified Heat Equation

We note that from the steady-state temperature equation A24 for the modified heat equation, K and f as well as h (through the formula $g = \frac{2(a+b)}{a\cdot b}h$) also appear as quotients ($\frac{f}{K}$ and $\sqrt{\frac{g}{K}}$). Hence, curve fitting will not produce unique values for them but only for their quotients. Using initial guesses K = 1000 W/(mK), f = -160000 W/m^2, and h = 0.5 W/(m^{2}K), ***fminsearch*** produced the estimated values K = 408.28 W/(mK), f = -150590 W/m^2, and h = 13.6 W/(m^{2}K). The thermal conductivity value of 408.28 W/(mK) is almost the same as those estimated from the original heat equation. However, the modified heat equation fit the experimental data much better than the original heat equation (**Figure 2**). The Newton cooling constant h = 13.6 W/(m^{2}K) is also within the range reported in the literature. The range values of h is 2.8-23 for still air and 11.3-55 for moving air (2). Finally, for the aluminum bar, we obtained from ***fminsearch*** K = 195 W/(mK), f = -206000 W/m^2, and h = 18.3 W/(m^{2}K). The Newton cooling constant is still within the range reported in the literature. However, aluminum thermal conductivity value is lower than the value for the copper bar indicating that the aluminum bar is not as efficient at conducting heat.

One possible extension of this study in the future is to design an experiment in which we have a perfectly insulated rod. However, such an experiment, which will allow us to verify the validity of the original heat equation, is very difficult to implement in real laboratory conditions.

MATERIALS AND METHODS

Experimental Data

The heat experiment (**Figure 4**) consists of a long copper and aluminum bar, respectively. The copper bar is 70 cm length with a 1 cm^2 cross section, with round circular holes drilled about 4 cm apart along the bar to accommodate a series of 15 thermocouples placed along the bar to be used for temperature measurement. The heating element is a soldering iron encapsulated cylindrical heater of 30 W. Although factors such as cross-sectional area of the rod, the distance between two adjacent holes, and the dimension of the holes can indeed affect the result, their effect is much smaller than that of heat loss, which is the focus of our

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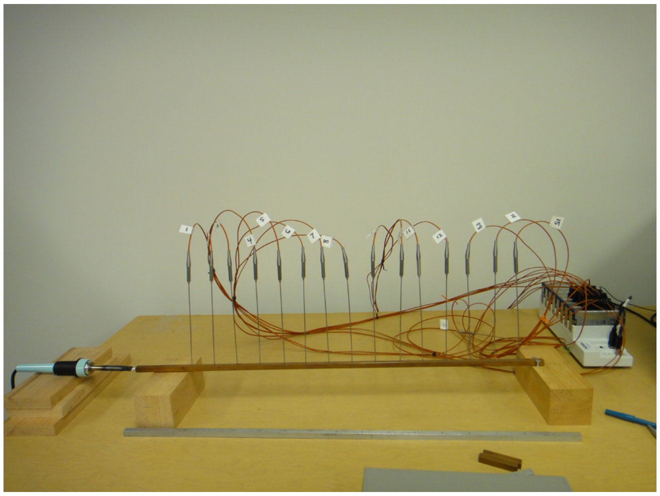

Figure 4: Experimental set-up. The experiment was carried out on a square metal bar of about 75 cm length and 1 cm^2 cross section, with holes (in which thermocouples can be inserted) drilled about 4 cm apart along the bar. We used both copper and aluminum bars in the experiment to study the properties of different metals and how they affect the heat conduction. The heating element used was a soldering iron encapsulated cylindrical heater of 30 W. The thermocouples were fit tightly into the drilled holes to ensure that thermal equilibrium between the rod and the thermocouple was established quickly.

experiment. In addition, we believe the power of the heater will only affect how long we must wait for the temperature of the rod to reach a steady state.

Curve Fitting

To illustrate how well steady-state temperature equations A23 and A24 fit the experimental data, we determined three physical parameters in the model (*f*,*K*,*h*) to give the best curve fitting of the model to the data. All computations were carried out using MATLAB (MathWorks). MATLAB is a multi-paradigm programming language and numerical computing environment that allows for data visualization as well as complex functions and computational algorithms to be implemented in a relatively simple manner. We used ***fminsearch*** function from MATLAB to find the three physical parameters in the model to obtain the best curve fitting. The best curve fitting is achieved by determining the physical parameters (*f*,*K*,*h*) to minimize the sum of squared residual errors between the data and the model equations, equation A23 or A24 as described by:

$$\min_{f,K,h} J = \sum_{i=1}^{15}\left(u_i^{data} - u_i^{m}\right)^2 \quad \textbf{(3)}$$

where denotes the x_i location of the thermocouple along the length of the rod associated with the ith thermocouple, u_i^{data} is the steady-state temperature data at the location x_i, and u_i^m is the steady-state temperature calculated from the model equations A23 or A24 at the location x_i. ***fminsearch*** uses the Nelder-Mead algorithm to find the minimizer of a function of multiple variables. Nelder-Mead is a pattern search optimization algorithm that does not require the gradient information to find a local minimum of a function. The algorithm works by using a shape function called a simplex. At each iteration, the vertex with the worst function value is replaced with another point with a better value. The

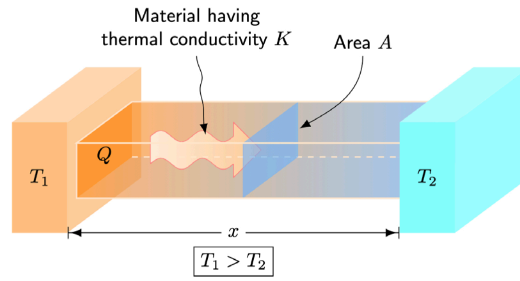

Figure 5: Heat conduction in a solid. The temperature T_1 at the left end of the material is assumed to be larger than T_2 at the right end so that heat is flowing from left to right. Q is the net heat transfer at the left end.

user provides an initial guess for the parameters (*f*,*K*,*h*) and ***fminsearch*** will iteratively update the parameters so that the function *J* in Equation (3) decreases in values at each iteration. ***fminsearch*** will terminate the iteration when the tolerances on the function *J* and parameter values between two successive iterations are less than or equal to 10^{-4}.

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Appendix

Mathematical Modeling

To investigate the rate of heat transfer by conduction in the copper bar, we first consider Fourier's law of heat conduction. This law demonstrates the influential factors in the heat transfer as shown in the following equation (2):

$$q = \frac{dQ}{dt} = -KA\frac{du}{dx} \tag{A1}$$

where Q is the net heat transfer in Joules, t is the time taken in seconds, u is the temperature in Kelvin (or ℃), x is the length between the two ends in meters, K is the thermal conductivity in $\frac{\text{W}}{\text{m·K}}$, A is the cross-sectional area normal to the direction of heat flow in m^2, and q is the rate of heat transfer and is given in units of $\frac{\text{Joule}}{\text{sec}}$ (**Figure 5**) In addition, we make the following standard assumptions:

1) The heat transfer inside the copper rod is solely by conduction;
2) The heat transfer is along the axis x, or the length of the copper rod, from left to right;
3) The temperature is uniform over a cross section of the rod;
4) The rod is perfectly insulated, meaning that there is no heat escaping through the side of the rod.

Based on these assumptions, we seek an equation to describe the temperature at any time *t* and any point *x* in the rod. The copper rod can be divided into several small segments, and for each segment that starts at one end x and ends at the other end $x + \Delta x$, if we suppose that the length of the segment ($\triangle x$) is infinitely small, then it would become almost as a surface, or a cross section. Subsequently, the net rate of heat accumulation equals the rate of heat input minus the rate of heat output.

Let the function $u\ (t, x)$ denote the temperature at time *t* and location *x* of the rod. Let $H\ (t, x)$ denote the heat (energy) transferred, which is proportional to the mass and the temperature: $H\ (t, x) = c_p m\, u(t, x)$ (Joule) where c_p is the specific heat (Joule/(kg.K)and m (kg) is the mass. Then, for a small segment of the copper rod that starts at one end x and ends at the other end $x + \Delta x$, the rate of heat accumulation of a certain segment can be expressed as:

$$\frac{dH}{dt} = \frac{d}{dt}\left(c_p m\, u(t, x)\right). \tag{A2}$$

Assuming no heat is generated inside the small segment $(x, x + \Delta x)$, the rate of heat accumulation is equal to the rate of heat input at x minus the rate of heat output at $x + \Delta x$, where the rate of heat input (according to Fourier's law of heat conduction) of the segment is given by:

$$q = -KA\frac{du}{dx}(t,x) \qquad \text{(A3)}$$

and the rate of heat output is given by:

$$q = -KA\frac{du}{dx}(t,x + \triangle x). \qquad \text{(A4)}$$

That is,

$$\frac{d}{dt}\left(c_p m\, u(t,x)\right) = -KA\frac{du}{dx}(t,x) + KA\frac{du}{dx}(t,x + \triangle x). \qquad \text{(A5)}$$

Since $\rho = density = \frac{mass}{volume} = \frac{m}{A\Delta x}$, we have $m = \rho A \Delta x$. Substituting this expression for m and dividing both sides of the equation by A, we obtain

$$\rho\Delta x\frac{d}{dt}\left(c_p\, u(t,x)\right) = -K\frac{du}{dx}(t,x) + K\frac{du}{dx}(t,x + \Delta x). \qquad \text{(A6)}$$

Dividing both sides by Δx, we get

$$\rho\frac{d}{dt}\left(c_p\, u(t,x)\right) = K \cdot \frac{\left[\frac{du}{dx}(t,x+\Delta x) - \frac{du}{dx}(t,x)\right]}{\Delta x}. \qquad \text{(A7)}$$

Taking the limit as $\Delta x \to 0$ yields

$$\rho\frac{d}{dt}\left(c_p\, u(t,x)\right) = K\lim_{\Delta x \to 0}\frac{\left[\frac{du}{dx}(t,x+\Delta x) - \frac{du}{dx}(t,x)\right]}{\Delta x} = K\frac{d^2}{dx^2}\, u(t,x). \qquad \text{(A8)}$$

Now, assuming c_p is constant, the above equation can be rewritten as

$$\frac{d}{dt}u(t,x) = \left(\frac{K}{\rho c_p}\right) \cdot \frac{d^2}{dx^2}\, u(t,x). \qquad \text{(A9)}$$

The coefficient $\alpha = \frac{K}{\rho c_p}$ is known as the thermal diffusivity in units of $\frac{\text{m}^2}{\text{sec}}$, which represents the ability of a material to conduct heat relative to its ability to store heat. When the coefficient is small, the material is less conductive, and when the coefficient is large, the material serves as a better conductor. Equation A9 is known as the one-dimensional heat equation. In this paper, we refer to equation A9 as the original heat equation.

Since all the data collected from the laboratory is steady-state data (reaching dynamic equilibrium), $\frac{du}{dt} = 0$. Hence,

$$\alpha \cdot \frac{d^2u}{dx^2} = 0 \;\; => \frac{d^2u}{dx^2} = 0. \qquad \text{(A10)}$$

Integrating the equation above twice, we obtain the function $u(x)$: $u(x) = c_1 x + c_2$. To determine the constants c_1 and c_2, we assume that at the left end of the rod where the soldering iron was used to input heat to the copper rod, the temperature can be described in terms of heat flux f, which is the rate of heat transfer per unit cross-sectional area and is given by $K \cdot \frac{du}{dx}(0) = f$, which implies $\frac{du}{dx}(0) = c_1 = \frac{f}{K}$. At the other end of the copper rod (**Figure 4**), we can either assume that the temperature of the rod is the same as the room air temperature or the temperature measured by the 15th thermocouple (the one closest to the end of the copper rod). The latter is physically more reasonable since the temperature of the rod is higher than the room air temperature due to the heat input from the soldering iron. Hence, $u(L) = u_{15}^{data}$ which implies $u(L) = c_1 L + c_2 = u_{15}^{data} \Longrightarrow c_2 = u_{15}^{data} - c_1 L$. Hence, the steady-state temperature in the copper rod is given by the following expression:

$$u(x) = \frac{f}{K} x + (u_{15}^{data} - \frac{f}{K} L). \quad \text{(A11)}$$

However, such a simple liner equation for the steady-state temperature derived from the original heat equation clearly does not take into consideration some important physical factors. In fact, from the experimental set-up (**Figure 4)**, some of the heat is lost to the air by heat convection. Using Newton's law of cooling, the rate of heat transfer by convection is proportional to the temperature difference, $h \cdot A\,(u - u_{air})$, where h is the Newton cooling constant and u_{air} is the room air temperature (2). Taking heat loss into consideration, the rate of heat accumulation in the small segment of the rod equals the rate of heat input minus the rate of heat output as well as the rate of heat loss by convection to the air (including the top, bottom, and the other two sides of the rod but excluding the front and back ends of the rod, which are relatively small). If we assume that the width of the rod is a and the thickness of the rod is b, the heat loss at the top and the bottom (of a small segment) equals $-2ha\Delta x(u(t,x) - u_{air})$, and the heat loss at the other two sides of the rod equals $-2hb\Delta x(u(t,x) - u_{air})$. The modified equation can be expressed as:

$$\frac{d}{dt}\left(c_p \rho A \Delta x\, u(t,x)\right) = -KA\frac{du}{dx}(t,x) + KA\frac{du}{dx}(t,x+\Delta x) - 2ha\Delta x(u(t,x) - u_{air}) - 2hb\Delta x(u(t,x) - u_{air}). \quad \text{(A12)}$$

By dividing both side by $A\Delta x$, we obtain:

$$\frac{d}{dt}\left(c_p \rho\, u(t,x)\right) = K\frac{\left[\frac{du}{dx}(t,x+\Delta x) - \frac{du}{dx}(t,x)\right]}{\Delta x} - \frac{(2ah+2bh)}{A}\,(u(t,x) - u_{air}). \quad \text{(A13)}$$

Taking the limit as $\Delta x \to 0$, we obtain the modified heat equation:

$$\frac{d}{dt}\Big(c_p\rho\, u(t,x)\Big) = K \cdot \frac{d^2}{dx^2}u(t,x) - \frac{2h(a+b)}{ab}\,(u(t,x) - u_{air}). \qquad \text{(A14)}$$

Equation A14 is the modified heat equation that takes into consideration the heat loss due to convection. The steady state condition, where temperature is a function of x only, can also be derived from the following equation by setting $\frac{du}{dt} = 0$:

$$K \cdot \frac{d^2u}{dx^2} - \frac{2h(a+b)}{a \cdot b} \cdot (u(x) - u_{air}) = 0, \qquad \text{(A15)}$$

$$Ku'' - \frac{2(a+b)}{a \cdot b}h \cdot u(x) = -\frac{2(a+b)}{a \cdot b}h \cdot u_{air}. \qquad \text{(A16)}$$

To simplify the equation, we introduce g to denote the $\frac{2(a+b)}{a \cdot b}h$, and the equation becomes:

$$Ku'' - g \cdot u(x) = -g \cdot u_{air}. \qquad \text{(A17)}$$

The solution to the modified steady-state heat equation is given by:

$$u(x) = c_1 e^{-\sqrt{\frac{g}{K}} \cdot x} + c_2 e^{\sqrt{\frac{g}{K}} \cdot x} + u_{air}. \qquad \text{(A18)}$$

Here, the constants c_1 and c_2 are determined from the conditions $K \cdot \frac{du}{dx}(0) = f$ and $u(L) = u_{15}^{data}$ as was done earlier for the original steady-state heat equation. To this end, we have:

$$\frac{du}{dx}(0) = -c_1\sqrt{\frac{g}{K}} + c_2\sqrt{\frac{g}{K}} = f/K, \qquad \text{(A19)}$$

$$u(L) = c_1 e^{-\sqrt{\frac{g}{K}} \cdot L} + c_2 e^{\sqrt{\frac{g}{K}} \cdot L} + u_{air} = u_{15}^{data}. \qquad \text{(A20)}$$

Solving this set of equations for c_1 and c_2, we obtain:

$$c_2 = \frac{\sqrt{\frac{g}{K}}(u_{15}^{data} - u_{air}) + \frac{f}{K}e^{-\sqrt{\frac{g}{K}} \cdot L}}{\sqrt{\frac{g}{K}}\left[e^{\sqrt{\frac{g}{K}} \cdot L} + e^{-\sqrt{\frac{g}{K}} \cdot L}\right]} \qquad \text{(A21)}$$

$$c_1 = c_2 - \frac{f}{K}\sqrt{\frac{K}{g}}. \qquad \text{(A22)}$$

In summary, we have derived two mathematical expressions for the steady-state temperature in a copper rod. The first equation was obtained by assuming that we have perfect insulation and is given by:

$$u(x) = \frac{f}{K}x + (u_{15}^{data} - \frac{f}{K}L). \qquad \text{(A23)}$$

The second steady-state temperature equation was obtained by removing the perfect insulation assumption since it is very difficult to satisfy in real physical experiments. This equation is given by:

$$u(x) = \left\{ \frac{\sqrt{\frac{g}{K}}(u_{15}^{data} - u_{air}) + \frac{f}{K} e^{-\sqrt{\frac{g}{K}} \cdot L}}{\sqrt{\frac{g}{K}} \left[e^{\sqrt{\frac{g}{K}} \cdot L} + e^{-\sqrt{\frac{g}{K}} \cdot L} \right]} - \frac{f}{K}\sqrt{\frac{K}{g}} \right\} e^{-\sqrt{\frac{g}{K}} \cdot x} + \frac{\sqrt{\frac{g}{K}}(u_{15}^{data} - u_{air}) + \frac{f}{K} e^{-\sqrt{\frac{g}{K}} \cdot L}}{\sqrt{\frac{g}{K}} \left[e^{\sqrt{\frac{g}{K}} \cdot L} + e^{-\sqrt{\frac{g}{K}} \cdot L} \right]} e^{\sqrt{\frac{g}{K}} \cdot x} + u_{air}, \quad \text{(A24)}$$

where $g = \frac{2(a+b)}{a \cdot b} h$. In the above two steady-state equations, the physical parameter f denotes the heat flux from the soldering iron, K denotes the thermal conductivity, and the parameter h is the Newton cooling constant. The values for these parameters were determined from the least-squares curve fitting problem using experimental data.