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Article

An Analysis of the Mathematical Accuracy of Perspective in Paintings

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SUMMARY

Art is a great human endeavor and artists, since time immemorial, have tried to capture the beauty of what lies around us in their canvases. But does art accurately depict real-world objects in terms of their sizes and proportions? A particular area where artists have been challenged is in the depiction of the three-dimensional world on a two-dimensional canvas. This is known as perspective. Mathematicians have done a lot of work to understand perspective and this field of mathematics is known as projective geometry.

Our hypothesis was that there are mathematical inaccuracies of perspective in artists' paintings that we are unable to detect with our naked eyes. Errors of perspective may be more tolerated as the distance from the eye to the object in the painting increases. We wanted to understand the degree of mathematical inaccuracy in several famous paintings and then draw conclusions about the limits of our perception of depth.

We have used a mathematical method from the world of projective geometry, which is called cross-ratio, to analyze paintings for accuracy of perspective. Cross-ratio is measured by using four fixed points on a straight line as reference; this value is always the same irrespective of the viewing angle or the viewing distance. We took structures/buildings in paintings as our four points and compared the painting's cross-ratio to that of a photograph of the same building. For our research we have taken three famous paintings made by different artists and measured their accuracies. From our analysis we concluded that there are significant errors in perspective in these paintings and the errors increase as the structures in the paintings recede from the viewer.

INTRODUCTION

Perspective is the technique to create the illusion of three-dimensional (3D) objects on a two-dimensional (2D) plane in art. The basic idea of perspective is that objects of a certain size decrease in size as they move farther away from the point of sight. Another technique employed by artists is called foreshortening. Foreshortening is the illusion that objects appear distorted if they are placed at an angle to the observer. For example, a circle may appear as an ellipse when seen from the side. Additionally, professional artists use the technique of vanishing points in their paintings. There may be one or more vanishing points on a plane. We treat these points as infinity, and all parallel lines meet at these point(s). The placement of objects on the painting with respect to these points and their relative sizes determines the visual accuracy of perspective (1–3). Foreshortening and the technique of vanishing points are complementary to each other and are important tools for artists (4).

Brunelleschi was the first to fully understand the importance of perspective and was able to incorporate the mathematical theory of perspective in his paintings in the fifteenth century. He claimed to discover a method to find out an object's dimensions in the picture by finding out the object's distance from the vanishing points. He applied his technique and created both paintings and architectural works. He is famous for creating the Duomo, a famous church in Florence. But many of his paintings that illustrate his techniques of perspective using mathematics have been lost (5,6). So, we asked if there is a way to measure the accuracy of perspective in paintings?

We were able to apply a simple but powerful method that could answer this question. From a branch of mathematics known as projective geometry we found a technique called cross-ratio (7). Cross-ratio is the ratio of four points lying on a line. The distances between these points are inserted into a formula to obtain the cross-ratio of those points.

If you draw lines starting from one of the vanishing points intersecting all four of our special points individually, extend them, and make a different line at any angle, you will get a set of four new points whose cross-ratio will be exactly equal to the previous one (6). As an example, the cross-ratio of the points A, B, C, and D will be exactly same to the cross-ratio of points A', B', C', and D' **(Figure 1A)**.
Cross-ratio is measured by evaluating the following formula:

$$\text{Cross-ratio of points (A, B, C, D)} = \frac{AC \times BD}{BC \times AD}$$

We infer from the above example that we get the same cross-ratio even if we scale the distances between points or change the angle of the line on which our points lie. Therefore, the angle of sight or the scale of the picture has no effect on the cross-ratio of any four points on a line (8).

The applications of the technique of cross-ratio are wide ranging. Scientists have proposed that accidents can be reconstructed by measuring the cross-ratio using a single camera placed at an angle to the collision (9). Since cross-ratio is invariant with respect to the angle of view, the speed of colliding vehicles can be accurately determined. This technique may be very useful in the field of forensic science. The method of cross-ratio can also be very important in the study of architecture, and it has been used to study the visual perception of old Roman buildings (10).

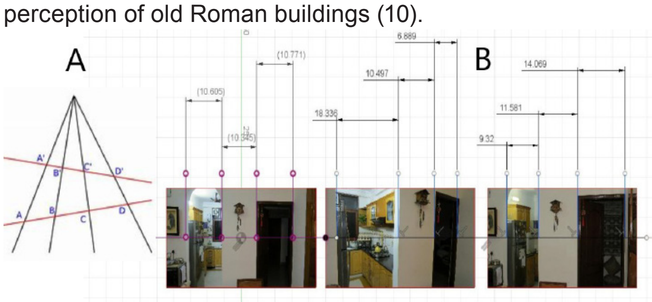

Figure 1: A) An illustration of the method of cross-ratio. B) The same four points in a room seen from different angles and distances. The cross-ratio of the four points is invariant.

We have used the method of cross-ratio very effectively in our study to analyze how accurate artists have been in their paintings using perspective. But the question remains, which four points in the painting should be analyzed?

Many artists make pictures of buildings or structures of great importance. We used the fact that several of those buildings/structures have survived to this date. Therefore, we have assigned our four points to lie on the buildings. Pillars have been the choice of placement for our points, as they are cylindrical and are typically of constant width to maintain the symmetry of the building. By comparing the cross-ratios of any four points on a building depicted in a painting to the cross-ratio of the same points on the picture of the real building, we could determine the accuracy of the painting to a great degree of detail.

Our hypothesis was that there are mathematical inaccuracies of perspective in the artists' paintings that we are unable to detect with our naked eyes. Errors of perspective are likely to be overlooked as the distance from the eye to the object in the painting increases. We wanted to understand the degree of mathematical inaccuracy in several famous paintings and then draw conclusions about the limits of human perception of depth.

The paintings we analyzed were *The Doge's Palace* and *Riva degli Schiavoni* by Giovanni Canal, *St. Peter's Basilica* by Giovanni Paolo Panini and *The Wedding at Cana* by Paolo Veronese.

RESULTS

Paintings by Giovanni Canal, Giovanni Paolo Panini, and Paolo Veronese were analyzed in order to find out if their depictions of buildings were mathematically accurate.

Before we analyzed the paintings of the masters, we studied the usefulness of the method of cross-ratio by taking a practical example. For this illustration we took photos of the interior of a building at different angles and distances **(Figure 1B).** We then chose four points on a straight line and measured their cross-ratio.

We assigned the end points of the door frames as our special points and named them A, B, C, and D.

We made a Python program to calculate the cross-ratio of the pictures by inserting the distances between the special points. We used a software that enabled us to zoom in the picture and measured the distances accurately and all the pictures were of very high quality, so the details were clear even when zoomed in.

We found out that the cross-ratio of the four points in the three parts measured were 1.32, 1.34, and 1.33 **(Figure 1B).** Thus, the cross-ratios varied just by 1-2% of each other, which was within the uncertainty of measurement in calculating distances.

Therefore, we concluded that the method of cross-ratio is a very reliable and precise method to measure how the relative distances between points change as the image is rotated or resized. We demonstrated that the cross-ratio of any four points on an image is invariant even if we change the angle of observation or distance from the object to the point of observation. Using this method, we could now proceed to measure the precision of several paintings.

Giovanni Antonio Canal

Giovanni Antonio Canal is considered one of the most famous and influential Italian painters in history. He is known for his exceptional interest in painting famous buildings and bridges in Venice and other cities of Italy (11). He painted over 600 paintings including the famous T*he Doge's Palace and Riva degli Schiavoni, The Grand Canal from the Palazzo Vendramin-Calergi towards S. Geremia, View of the Grand Canal, Piazza San Marco* and *The Molo from the Basin of San Marco* (12). He was known for the accuracy and precision in replicating the buildings and all their details in his paintings.

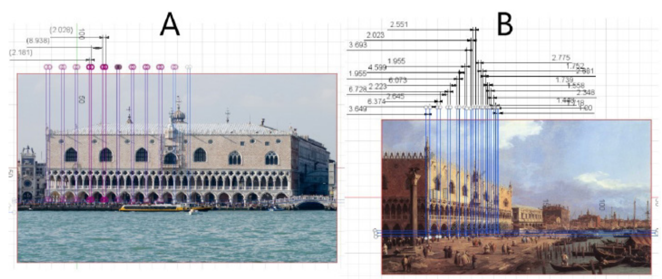

Figure 2: A) A photograph of the *Doge's Palace* in Venice. The pillars are marked to calculate the cross-ratios which will be used to compare with the painting. B) Painting of *Doge's Palace* by Canal on which we have marked the distances between the pillars. The distances are used to calculate the cross-ratio of the points taken four at a time.

We looked at the accuracy of *The Doge's Palace* and *Riva degli Schiavoni* by comparing its cross-ratio to that of the real building using a photograph **(Figure 2).** By using the software, Autodesk Fusion 360 (13), we marked points on each of the pillars in the Canal painting and measured the distances between them **(Figure 2B)**. This software enabled us to zoom into the picture to measure the distances accurately. In the photograph of Doge's Palace we marked the pillars that we marked in the painting as reference **(Figure 2A)**. We assigned the same special points to each of the pillars of the palace and measured the distances between them. Since we needed four points to measure the cross-ratio, we combined two points of one pillar with two of the subsequent one.

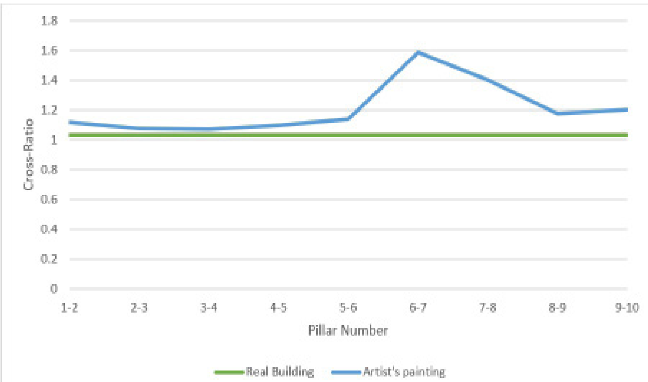

Figure 3: Comparison of cross-ratios of the real building (Doge's Palace) to that of the painting by Canal.

We showed that there is a significant variation of cross-ratios in the artist's painting **(Figure 3)**. The artist was very accurate in the first few pillars and the cross-ratio of the dimensions of his pillars is quite close to that of the real building, but after the fifth pillar the difference in the cross-ratios becomes considerable. The artist was most inaccurate in column six, where he had an error of 47 %. In pillar three the artist was most accurate, where the error was only 3%. On average the error in the painting compared to the actual building was 16.5% **(Figure 3)**.

Giovanni Paolo Panini

Giovanni Paolo Panini was a famous Roman painter and architect in the eighteenth century. He painted many different vistas of Rome, from its architecture to its landscape. He painted many paintings of different Roman monuments including *The Pantheon, A Capriccio of the Roman Forum, The Colosseum, St. Peter's Basilica*, and many more (14). His legacy remains in his precision of replicating Roman buildings on his canvases. We examined the accuracy of his painting, *St. Peter's Basilica*.

We assigned special points to the pillars on the right in the Panini painting and calculated their cross-ratio **(Figure 4)**, as well as those in the real picture taken in *St. Peter's Basilica* **(Figure 5)** (15). We have shown the dimensions of the points on the pillars in the picture which helped us find the cross-ratios. These points correspond to the ones we used to calculate the cross-ratios of Panini's painting.

Thus, we saw that the artist was most inaccurate in the dimensions of pillar three, where his error was 16.2% **(Figure 6)**. The artist was slightly more accurate in pillar 1 in which the error was 15.6%. When compared to the real building, the painting has on average 16% error in perspective. But we could see that the cross-ratio jumped significantly from pillar two to pillar three. This variation was about 33% between pillar 2 and pillar 3 **(Figure 6)**. We showed from this analysis that Panini was fairly inconsistent and inaccurate in his painting.

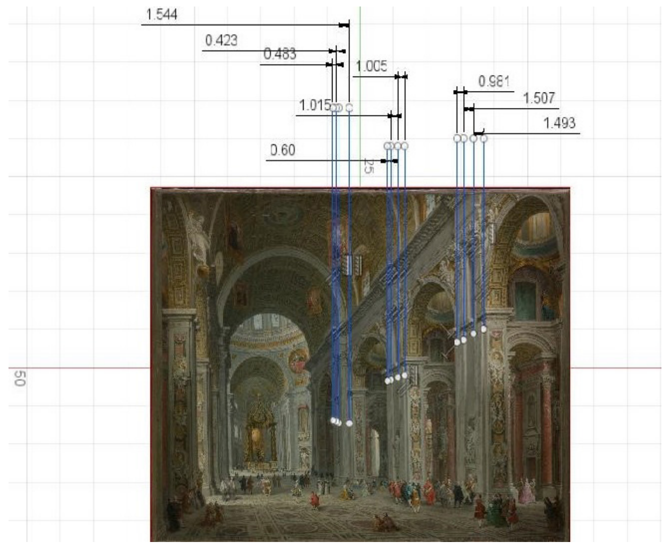

Figure 4: *St Peter's Basilica* painted by Panini. We added the lines to measure distances to calculate cross-ratios.

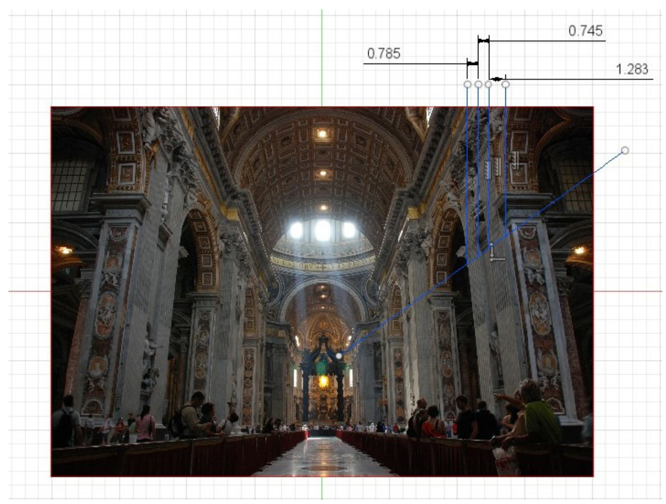

Figure 5: A photograph of *St Peter's Basilica.* We added the lines to measure distances to calculate cross-ratios.

Paolo Veronese

Paolo Veronese was one of the most famous painters in the sixteenth century. He was famous for making large-scale and grand paintings which had a lot of detail (16). Some of his famous works are *Venus and Adonis*, *The Wedding at Cana, The Vision of St. Helena* and *The Resurrection of Christ*. We wanted to analyze the accuracy of *The Wedding at Cana* (17). This painting is the largest painting in the Musée du Louvre in Paris. We used the technique of cross-ratio described earlier to determine the accuracy of the artist's painting. We have depicted the painting and the points that we used to calculate the cross-ratios and the distances between them **(Figure 7)**.

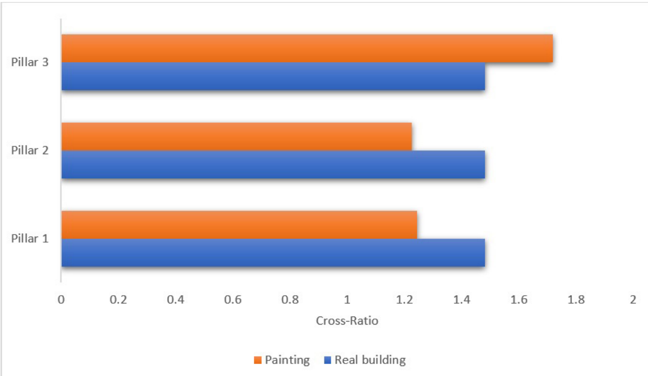

Figure 6: Cross-ratios of Panini's painting compared to a photograph of *St Peter's Basilica*.

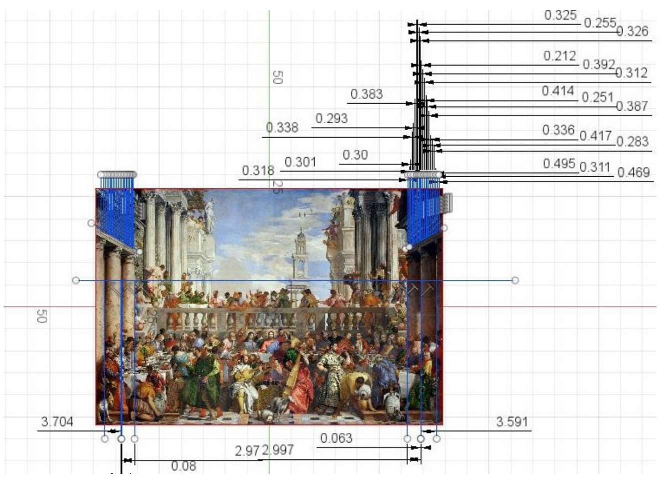

Figure 7: *Wedding Feast at Cana*. We have marked the points we used in calculating the cross-ratios.

The building depicted in the painting is not real, but is based on the imagination of the artist. We assumed that the artist completely mirrored the marble designs on both sides, and therefore the cross-ratio should remain same. We calculated the cross-ratio of the repeated design on top of the columns of the building at the left and the cross-ratio of the repeated design on top of the columns of the building at the right. We concluded from this data that the maximum variation between the two similar designs on both sides was 26%, and on average the cross-ratios of both sides agreed with each other by 95% **(Figure 8)**. Therefore, in this painting we could also see inconsistency in the cross-ratio on both sides.

DISCUSSION

With the analyses of the cross-ratio in three paintings we showed that all of these artists became less accurate in perspective as their work receded to one of the vanishing points. All the above-mentioned paintings show inaccuracies of approximately 3% to 16% in the foreground, and it becomes worse as the objects in the painting recede into the horizon. All the paintings were painted by popular artists and have been visited by millions of people. One finds it strange that most viewers haven't noticed inaccuracies in these paintings. We conclude that the human eye ignores inaccuracies to a certain limit, and in this study we have shown that even inaccuracies of more than 15% in some cases are ignored. However, our study could not find the limit at which the human eye can detect this inaccuracy. We suggest that a more detailed study on this is needed to establish this limit accurately. However, it is clear from our analysis that as the structures in the building recede into the vanishing point the inaccuracies might increase and remain undetected by the artist and the viewers. As the distance from the eye increases, the details of a structure's features become so minute and blurred that the human eye completely ignores the errors in perspective.

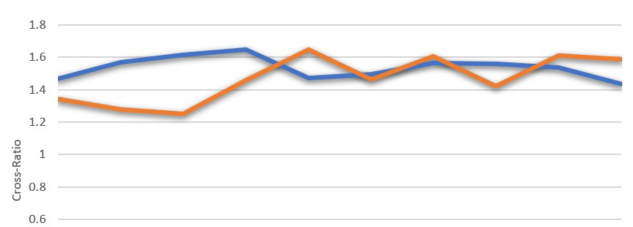

Figure 8: Comparing Cross-ratios of similar patterns in the painting, "*The Wedding at Cana*".

The mathematical idea of cross-ratio is very useful as it is invariant with respect to the angle of sight. However, we believe that human perception will be impacted as the angle of the viewer changes and will also depend on other factors like size of structures and distance. Therefore, we think that this could be an idea for a future work in which surveys can be conducted to see how people view inaccuracies as buildings are rotated by various angles and are zoomed in/out.

Another idea of a potential future research project could be to rank some selected popular paintings by the accuracy of perspective using the methods presented in this paper. We could then conduct a survey by a set of people and ask them to rank the same paintings by how accurate they appear visually. We could then compare both rankings to find out the correlation between human visual perception and the mathematical accuracy of paintings.

Mathematics is a very useful tool to help us make sense of the world around us. Art is an expression of human emotion and a desire to capture the beauty of the outside world. But artists can make mistakes in capturing the external world because of the limits of what can be seen and perceived by the naked eye. We can use mathematics to find out these mistakes and correct them. Artists can use the techniques of projective geometry when making rough sketches of their paintings so that when the final painting comes out, the results are more accurate and more visually appealing and appear more realistic. In this paper we have successfully used mathematics to find out flaws of perspective that artists tend to make in their work.

MATERIALS AND METHODS

For this research, famous paintings which depicted monuments were selected. The reference image, which could be an actual picture of the same monument, was also searched online. High pixel density images (of at least 300 DPI) were chosen as these could be zoomed in without getting blurred. Once the paintings were selected, they were loaded into Autodesk Fusion 360 software. These images were zoomed in on and prominent structures of the buildings in the images were chosen. Pillars or other symmetric objects which were repeated in the structure were selected. We knew that the method of cross-ratio required four points, but a pillar gives us only two (the left edge and the right edge). So, to get the four points we combined two adjacent pillars in our calculations for the first painting (Canal). In the second painting (Panini), each pillar has four points as each pillar had substructures. Therefore, we did not need to combine pillars. Similarly, in the third painting (Veronese), the patterns on top of the pillars gave us many choices from which we could choose our four points from. Once the images were loaded into the software and the structures to be analyzed chosen, the line tool was used to draw lines on the edges of these structures. These lines were extended parallel to the y axis. The sketch dimension tool was used to find distances between the vertical lines. Once we had four distances, a simple Python program was used to find the cross-ratio of the points. These calculations were repeated for the row of pillars or other symmetric structures. Similar calculations were done for the real image and then the two sets of cross-ratios were compared to determine the accuracy of the painting. We defined the error rate in our calculations as follows:

$$\text{Error \%} = \left| \frac{\text{Cross ratio of four points in the real building} - \text{cross ratio of same points in painting}}{\text{cross ratio of the four points in the real building}} \right| \times 100$$

With respect to the choice of paintings there are some limitations using the technique of the cross-ratio. We required high quality images to be able to determine the cross-ratio accurately. The minimum resolution required for our analysis was 300 DPI. This enabled us to zoom in and accurately measure distances between objects without the details becoming blurred. Another requirement of this technique was that structures in the painting had to be repetitive while retaining their original size and be in a straight line. This enabled us to mark points in a straight line and measure their cross-ratio accurately and then compare this with the cross-ratio of the same structure in a photograph. Thus, this method will only apply to paintings that try to depict the details of buildings and similar structures accurately. This technique will not apply to paintings which are a part of art movements like impressionism, cubism, surrealism and so on, where the artist is not trying to capture the details of structures accurately, but takes artistic liberties with the dimensions of the structures.

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