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Article

Examining what causes perceived dissonance in musical intervals and the effect of timbre on dissonance

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SUMMARY

Dissonance is perceived when certain musical notes are heard simultaneously and create a harsh, clashing feeling. Consonance, the opposite of dissonance, is perceived as a pleasant harmony of musical notes. Timbre is the defining characteristic of sound which differentiates sounds from different sources. This study attempts to determine the cause of perceived dissonance and the effect of timbre on dissonance. We attempt to corroborate Hermann von Helmholtz's theory of temporal dissonance, which proposes that dissonance results from beat frequencies created by interference of harmonic overtones. Our procedure examines algebraic and graphical representations of sound waves of the nine standard musical intervals from the minor second to the octave, produced on piano, ranging from dissonant to consonant. No clear correlations were found between different quantitative metrics and dissonance. The study also compares graphs of intervals from a piano and a human voice, which have different timbres. We noted that the graphs of piano notes, which have a "harder"-sounding timbre, had higher ratios of concavity changes to beat period than the graphs of sung notes, which have a "softer" timbre. Further research is planned to study whether graphical characteristics can reveal qualities of timbre.

INTRODUCTION

Timbre is the tonal quality of sound. Unlike pitch or amplitude, timbre distinguishes a note on a piano from the same note on a violin. Timbre is perceived when the human brain interpretively fuses pure tones, or sinusoidal sound waves, that are at harmonic or nearly harmonic frequencies into one pitch known as a complex tone. The term "harmonic" refers to a harmonic series, a sequence of frequencies related to a fundamental frequency by whole number ratios (1). Each frequency in the sequence is a harmonic. Different instruments have different timbres due to the nature of their materials, such as strings or air columns. Although the same note or pitch played on each instrument has the same harmonic series, the relative loudness, or amplitudes, of the harmonics are different, thus creating distinct sound qualities for each instrument.

In music, combinations of two notes can be characterized by the musical intervals between the notes. These intervals are distinguished by comparing how dissonant they sound—that is, how harsh the notes are together. Consonance and dissonance can be described as the harmony and lack of harmony, respectively, between sounds and are easily and intuitively discernable by ear. Dissonant intervals are characteristically harsh-sounding, whereas consonant intervals are not. Hermann von Helmholtz's theory of temporal dissonance states that dissonance occurs when there is beating produced from harmonics of complex tones interfering with one another, with maximum dissonance produced at a beat frequency of around 35 Hz (hertz, cycles per second), and that dissonance is dependent on the magnitude of the interval between the pitches of the complex tones (2). Beat frequencies ranging from 10 to 60 Hz cause dissonance because the two frequencies producing it are too far apart to be registered as one pitch but too close to distinguish as distinct pitches (3). Because the relative amplitudes of harmonics for the same pitch played on different instruments are different, we hypothesized that the same intervals of pitches sounded from different instruments would be perceived as having different levels of dissonance.

Mathematician Joseph Fourier proved that all continuous functions can be broken up into an infinite number of sinusoidal waves (4). The Fourier Transform is able to break up a signal or waveform over time into the component sinusoids that make it up, representing the sinusoidal functions as graphs of amplitude against frequency (5). For complex tones, these frequencies are the fundamental frequency, which has the greatest amplitude, and its corresponding pure-tone harmonics. The study used the Fourier Transform to determine the equations for the most prominent pure tones of musical notes produced by different sources and analyze this information.

The aim of this study is to assess what causes perceived dissonance in musical intervals and whether timbre has an effect on dissonance. We hypothesized that Helmholtz's theory of temporal dissonance is correct. The experiment attempted to graphically and algebraically analyze dissonant intervals to corroborate Helmholtz's theory. This included examining the frequency of concavity changes in the graphs of sound waves as a possible determining characteristic of timbre, since adding sinusoidal functions of different frequencies and amplitudes results in different patterns of concavity. We also hypothesized that timbre affects perceived dissonance.

RESULTS

We tested Helmholtz's theory of temporal dissonance by creating equations to represent the sound waves of intervals being played on a piano. We first generated equations representing the sound waves produced when individual notes are played on the piano by using a Fast Fourier Transform (FFT) smartphone application (**Figure 1**) (6) to obtain frequency and amplitude values for the component pure tones of each note. We then added these equations together to create graphs that represent two notes being played simultaneously (**Figure 2**). Each graph is of a different

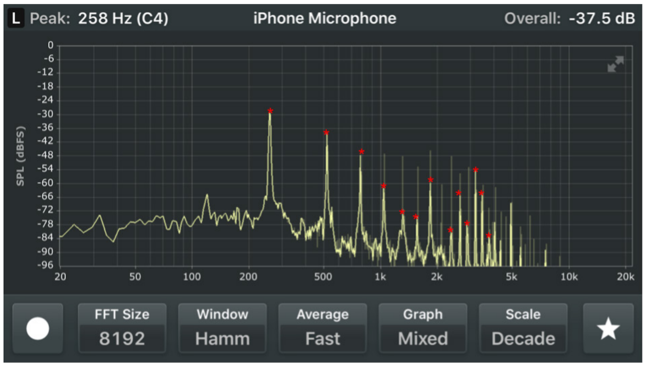

Figure 1. Fourier analysis of C4 on piano. On the piano used, the frequency of the note is 258 Hz rather than 261.626 Hz, which is likely due to the piano having gone slightly out of tune over time. The horizontal axis is frequency, in Hz, and the vertical axis is sound pressure level (SPL), in dBFS (decibels relative to full scale). dBFS is used in digital sound recordings, and full scale is the maximum loudness discerned by the device. dBFS describes the loudness of a sound relative to the full scale. Red asterisks denote the thirteen "peaks" for which frequency and SPL were recorded.

Table 1

Interval (most to least dissonant)	Number of beat frequencies between component frequencies between 10 and 60 Hz	Beat frequency (Hz)	Beat period	Number of concavity changes over t=0 to t=1/35	Number of concavity changes in 1 beat period	Concavity changes/beat period
C4-C#4	3	1.75E+01	5.71E-02	1.33E+02	2.59E+02	4.53E+03
C4-D4	4	3.33E+00	3.00E-01	1.33E+02	1.41E+03	4.68E+03
C4-G#4	2	4.58E+00	2.18E-01	1.79E+02	1.31E+03	6.02E+03
C4-D#4	6	5.00E+01	2.00E-02	1.19E+02	8.10E+01	4.05E+03
C4-A4	0	6.67E+00	1.50E-01	1.55E+02	8.27E+02	5.51E+03
C4-E4	2	1.67E+00	6.00E-01	1.52E+02	3.09E+03	5.15E+03
C4-F4	0	2.78E+00	3.60E-01	1.43E+02	1.85E+03	5.13E+03
C4-G4	2	3.33E+00	3.00E-01	1.44E+02	1.56E+03	5.20E+03
C4-C5	2	2.50E+00	4.00E-01	1.61E+02	2.24E+03	5.60E+03

Table 1. Table of calculated metrics for the nine musical intervals on piano. Intervals are ordered top to bottom from most to least dissonant according to Helmholtz's theory.

musical interval, with nine graphs in total representing the nine standard intervals between the minor second (C4-C#4) and the octave (C4-C5). All intervals had the lower note C4 and a higher note to produce the interval. For each interval, we calculated the number of beat frequencies between 10 and 60 Hz produced by the component frequencies of the notes in the interval, the observed beat frequency and period of the graph representative of the interval, the number of changes in concavity of the graph representative of the interval on $0 \leq t \leq 0.3$ s, and the ratio of the number of changes in concavity to the beat period of the interval

$$\frac{number\ of\ concavity\ changes\ in\ one\ beat\ period}{beat\ period},$$

in order to examine what causes perceived dissonance in musical intervals. To find the number of beat frequencies between 10 and 60 Hz produced by the component frequencies of the notes in the intervals, we calculated the beat frequencies produced by all combinations of component frequencies for each interval with the equation

$$beat\ frequency = |frequency_1 - frequency_2|.$$

We counted the number of beat frequencies which were between 10 and 60 Hz for each interval. We found approximate beat frequencies and periods for the graphs representative of musical intervals by viewing the graphs in an online graphing program (7) and estimating the number of full cycles over a known time interval. If cycles were unclear, we adjusted the time interval until cycles became clear. We then calculated the beat frequencies in Hz with the equation

$$beat\ frequency = \frac{number\ of\ cycles}{time\ in\ s}$$

in Google Sheets. The beat periods in seconds were calculated by taking the reciprocals of the beat frequencies. To find the numbers of concavity changes in the graphs of the intervals, we created equations for the second derivatives of the equations for each interval. We graphed the second derivative equations and counted the number of horizontal axis crossings over the interval $0 \leq t \leq 0.3$ s and over one

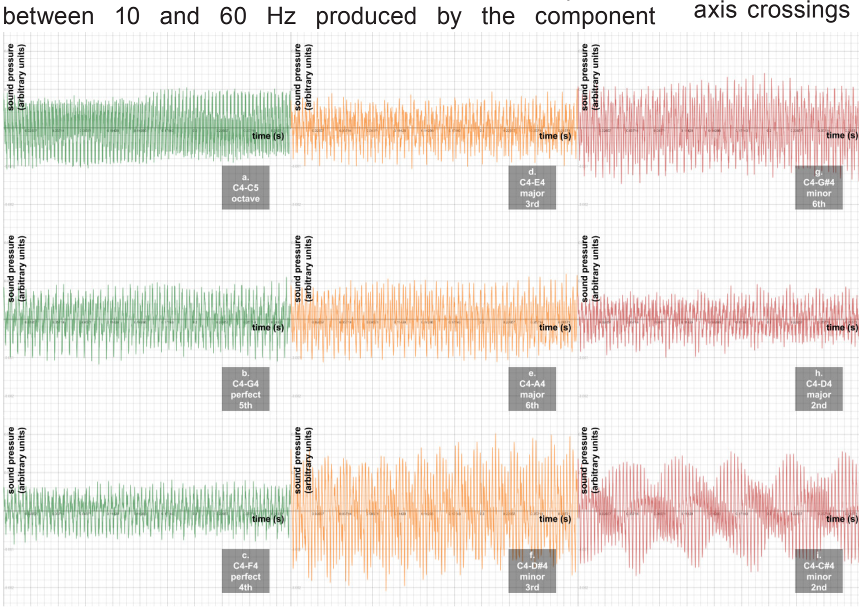

Figure 2. Graphs of intervals. These are not the original signals, but are equations representative of the sound waves created by summing sine waves with frequency and amplitude values derived from original signals. Played on piano ordered from least (top left; **a**) to most (bottom right; **i**) dissonant according to Helmholtz's theory. The horizontal axis of the graphs shows time elapsed in seconds, while the vertical axis shows the sound pressure except not to scale.

Table 2						
Interval	Number of beat frequencies between component frequencies between 10 and 60 Hz	Beat frequency (Hz)	Beat Period	Number of concavity changes over t=0 to t=1/35	Number of concavity changes in 1 beat period	Concavity changes/beat period
C4-C#4 sum	3	1.75E+01	5.71E-02	1.33E+02	2.59E+02	4.53E+03
C4-C#4 control	3	1.42E+01	7.06E-02	1.04E+02	2.58E+02	3.66E+03

Table 2. Table of calculated metrics for the representative equation of the C4-C#4 interval on piano created by summing representative equations for C4 and C#4 individually ("C4-C#4 sum", **Figure 1i**) and by recording FFT data for C4 and C#4 played simultaneously (**"**C4-C#4 control", **Figure 2).**

beat period ($0 \leq t \leq$ beat period), which equals the number of concavity changes in the original equations. The intervals were ordered from least to most dissonant based on Helmholtz's theory of dissonance (1). There appeared to be no clear correlation between dissonance and the number of beat frequencies between 10 and 60 Hz produced by the component frequencies of the notes in the intervals, between dissonance and the observed beat frequency of the graph for the interval, between dissonance and the number of concavity changes in the graph for the interval over the same time interval ($0 \leq t \leq 0.3$ s), or between dissonance and the ratio of concavity changes to the period of the observed beating for the equation of the interval (**Table 1**).

We conducted a control experiment to determine how similar the equations created by adding the equations for individual notes were to equations created for the sound wave of an interval. We created an equation representative of the sound wave produced by a C4 and C#4 played simultaneously on the piano. Visually, the control graph (**Figure 3**) is somewhat similar to the graph of the sum of the complex tone equations for C4 and C#4 (**Figure 2i**). We calculated the number of beat frequencies between 10 and 60 Hz produced by the component frequencies of the notes in the interval, the observed beat frequency and period of the graph representative of the interval, the number of changes in concavity of the graph representative of the interval on $0 \leq t \leq 0.3$ s, and the ratio of the number of changes in concavity to the beat period of the interval. There are approximately four and a quarter beats within the interval $0 \leq t \leq 0.3$ s in the control and approximately five and a quarter beats in the experimental results. Consequently, the beat frequency and period of the control graph and the summed graph (experimental) differ. The number of concavity changes over the same time interval and the ratios of concavity changes to beat periods also differ greatly between the control and sum graphs. However, the numbers of concavity changes in one beat period differ by only one, which is relatively small (**Table 2**). The numbers of beat frequencies between 10 and 60 Hz produced by component frequencies are equal. The discrepancies between the two graphs may be in part due to the C4 and C#4 not being played exactly simultaneously for the control graph, since the sum graph assumes the two notes are sounded at the exact same moment with no horizontal translation to account for a delay in the playing of one note. Despite the discrepancies, we chose to use the summation of individual notes played separately rather than equations of intervals played directly so that the bottom note in all intervals, C4, would have the same equation and serve as a controlled variable.

To determine what effect timbre has on dissonance, we compared a human voice to the piano. We created graphs of the notes C4 and C#4 and the C4-C#4 interval for both sources (**Figure 4**). We also created audio recordings of the C4-C#4 interval for both sources. Based on our perception by ear, the C4-C#4 interval sounded more dissonant on the piano than by voice. We concluded that timbre does affect perceived dissonance, affirming the hypothesis that the same interval sounded from different sources will be perceived as having different levels of dissonance. Individual notes sounded "harder" when played on piano than sung, which is a phenomenon difficult to describe in words.

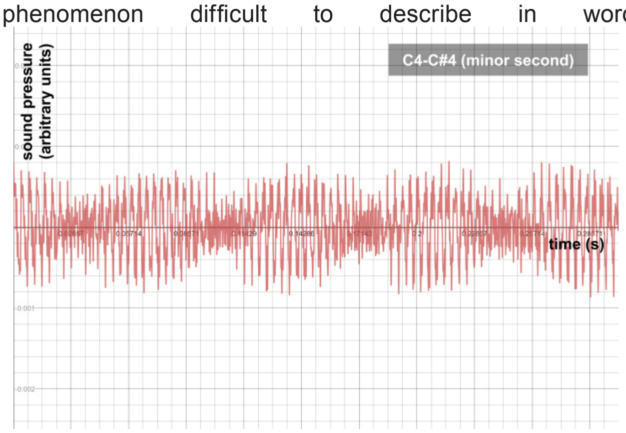

Figure 3. Graph of C4-C#4 interval (minor second) played on piano. This is not the original signal but rather an equation representative of the original signal created by summing sine waves with frequency and amplitude values derived from the original signal. The equation is not the sum of the equations for C4 and C#4.

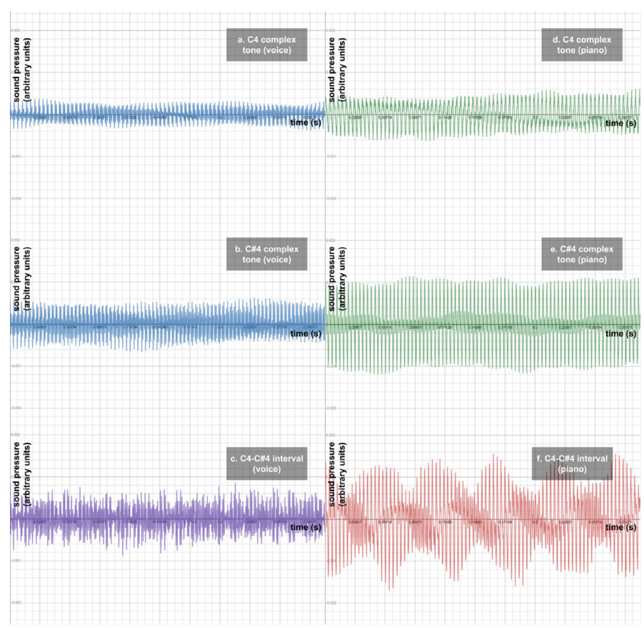

Figure 4. Graphs of C4 and C#4 notes. Not the original signals, but instead equations representative of the sound waves created by summing sine waves with frequency and amplitude values derived from original signals. C4-C#4 interval (minor second) created by adding the equations for the C4 and C#4 notes. Sung by voice (**a, b, c**) and played on piano (**d, e, f**).

Table 3							
Note/interval	Beat frequencies between component frequencies between 10 and 60 Hz	Frequency (Hz)	Period	Number of concavity changes over t=0 to t=1/35	Number of concavity changes in 1 beat period	Concavity changes/beat period
C4 piano	—	2.67E+02	3.75E-03	1.51E+02	2.10E+01	5.60E+03
C#4 piano	—	2.83E+02	3.53E-03	1.23E+02	1.70E+01	4.82E+03
C4 voice	—	2.50E+02	4.00E-03	6.60E+01	8.00E+00	2.00E+03
C#4 voice	—	2.75E+02	3.64E-03	1.38E+02	1.70E+01	4.68E+03
C4-C#4 piano	17, 32, 47	1.75E+01	5.71E-02	1.33E+02	2.59E+02	4.53E+03
C4-C#4 voice	17, 35, 56	5.67E+01	1.76E-02	1.18E+02	7.00E+01	3.97E+03

Table 3. Table of calculated metrics for C4 and C#4 notes. Not the original signals, but instead equations representative of the sound waves created by summing sine waves with frequency and amplitude values derived from original signals. C4-C#4 interval (minor second) created by adding the equations for the C4 and C#4 notes, on piano and by voice.

Using the data processing pipeline outlined above, we analyzed the individual C4 and C#4 notes for piano and voice, as well as the C4-C#4 interval by voice (**Table 3**). For both C4 and C#4, the piano note graph had a higher ratio of concavity changes to beat period than the same note by voice (**Table 3**). For C4, the number of concavity changes over the same time interval and the number of concavity changes in one beat period are also lower for voice than for piano. However, for C#4, the number of concavity changes over the same time interval is higher for voice than piano, while the number of concavity changes in one beat period is the same for both piano and voice. For the C4-C#4 interval, the number of concavity changes over the same time interval and the ratio of concavity changes to beat period are higher for piano, and the number of concavity changes in one beat period is higher for voice. The number of beat frequencies between component frequencies between 10 and 60 Hz is the same for piano and voice. The values of the beat frequencies are also relatively similar, which is expected because the harmonic frequencies for the same note should be the same regardless of the note source.

DISCUSSION

In this investigation, we hypothesized that Helmholtz's theory of temporal dissonance, which states that dissonance occurs when there is beating produced from harmonics of complex tones interfering with one another, with maximum dissonance produced at a beat frequency of around 35 Hz (hertz, cycles per second), and that dissonance is dependent on the magnitude of the interval between the pitches of the complex tones (2), is correct. We examined the nine standard musical intervals from the minor second to the octave. The musical interval from C4 to C#4 is a minor second because the notes are adjacent to each other on a piano with no key in between. To the ear, the two notes played simultaneously on the piano sound very dissonant because they are very close in pitch but perceptibly different, and there is a feeling of unresolved tension; according to Helmholtz's theory, the minor second is the most dissonant interval because the harmonics of the two pitches interfere to produce the most beat frequencies close to 35 Hz. The octave is the least dissonant (most consonant) interval according to Helmholtz's theory, with the frequencies of its two pitches related in a 1:2 ratio (one note's frequency is double that of the other). To the ear, two notes that are an octave apart sound like a high and low version of the same pitch. For example, imagine a grown man and a little girl singing the same note; most likely, they will be singing one or multiple octaves apart; to bystanders, there would be no dissonance created from the two sounds. There are also intervals beyond the octave, but these were not included in this study.

We attempted to corroborate Helmholtz's theory by conducting graphical and algebraic analysis of musical intervals played on a piano passed through a Fast Fourier Transform. We did not find any clear correlations between dissonance and the number of beat frequencies between 10 and 60 Hz produced by the component frequencies of the notes in the intervals, between dissonance and the observed beat frequency of the graph for the interval, between dissonance and the number of concavity changes in the graph for the interval over the same time interval ($0 \leq t \leq 0.3$ s), or between dissonance and the ratio of concavity changes to the period of the observed beating for the equation of the interval (**Table 1**). The results of the study do not conclusively support Helmholtz's theory of temporal dissonance.

We also hypothesized that timbre, the tonal quality of sound, has an effect on perceived dissonance. From listening by ear to the same interval (C4-C#4) from different sources (a piano and a human voice, in our case), we concluded that timbre does affect perceived dissonance. We examined the frequency of concavity changes in the graphs of sound waves as a possible determining characteristic of timbre, since adding sinusoidal functions of different frequencies and amplitudes results in different patterns of concavity. We repeated the graphical and algebraic analysis for intervals and individual notes played on a piano and sung by voice. Our results did not show consistent differences between notes and intervals sounded by voice and those played on piano. For the C4 note, the metrics for voice were consistently lower than the metrics for piano, but, for the C#4 note and C4-C#4 interval, some metrics were higher for voice, and some were higher for piano. To better examine whether the selected metrics are correlated to timbre, more notes and intervals should be tested, as well as more timbres.

In order to better draw conclusions regarding how sound waves interfere to produce dissonance and what could graphically signify dissonance, data and graphs for more different intervals should be gathered to determine a clearer trend. Intervals beginning with notes other than C4, as well as those beyond the octave, should be considered. As the control experiment indicated that the equations created by summing equations for individual notes differ from equations created from sounded intervals, it is a good idea to create equations from sounded intervals and conduct the same analyses and compare the results.

When we compared the same interval (C4-C#4) on piano and by voice, we found that the number of concavity changes over the same time interval and the ratio of concavity changes to beat period were higher for piano and that the number of concavity changes in one beat period was higher for voice. This may be related to timbre, but in order to investigate more thoroughly, it would be beneficial to create representative equations for complex tones produced by a variety of sources and examine whether there is a correlation between the ratio of concavity changes to beat period and "softness" in tonal quality. It would be interesting to investigate whether timbres can be discerned by graphical characteristics.

It should be noted that there are limitations in the data of this study. The selection of harmonic frequencies included in complex tone equations was done manually. At higher frequencies more distant from the fundamental frequency, it becomes harder to discern "peaks" by eye; thus, it is possible that some of the data taken were for frequencies other than harmonics, though these frequencies would be at lower amplitudes and therefore have less bearing on the shape of the complex tone graphs. For higher precision, one solution is to devise an algorithm to select "peaks" based on certain criteria. Another solution is to calculate the first twelve harmonic frequencies for each fundamental frequency and then refer to these values when recording FFT (Fast Fourier Transform) data. However, the calculated frequencies may not necessarily be "peaks". In this study, we decided to record data for "peaks" because "peaks" are at relatively higher amplitudes than their neighboring frequencies and would therefore have more bearing on the shape of the resulting complex tone graphs. It would be helpful to repeat the process of collecting values several times and use the averages of these values to create equations and graphs. Imprecision was also introduced by the FFT application rounding frequency values to integers and SPL values to one decimal point. A different FFT application or software could allow for more precision.

Based on hearing alone, we can conclude that timbre does affect perceived dissonance, with "softer" timbres producing less dissonance than "harder" timbres for the same interval. This relationship could be corroborated by listening to the same interval played by many different instruments whose timbres would be qualified beforehand. This would require a system of classifying and describing timbres as well as many people to rate timbres and degrees of dissonance. This study could be carried out using the Internet to collect ratings from volunteers. Furthermore, intervals of two different instruments (in which one instrument plays one note and another plays another note) could be examined, which would allow a more thorough understanding of the effect of timbre on dissonance. Understanding how timbre affects perceived dissonance can improve the creation of new music, allowing composers to refine the characters of their music with careful selection of instruments in consideration of how their timbres will interact. One interesting aspect is the consideration of timbre is computer sound synthesis, which can both replicate the sounds of real instruments and produce completely new sounds that are not found naturally. Computer-synthesized sounds, as often heard in contemporary pop music, are a collection of more timbres to be explored. As a continuation of this study, we plan to study the graphs of complex tones of different sources in order to determine whether there is a correlation between the ratio of concavity changes to beat period and "hardness" or "softness" in timbre.

In summary, the results of our study do not support Helmholtz's theory of temporal dissonance. We did not find a clear relationship between perceived dissonance in a musical interval and the number of beat frequencies between 10 and 60 Hz produced by the component frequencies of the notes in the interval, the observed beat frequency of the graph for the interval, the number of concavity changes in the graph for the interval over the same time interval, or the ratio of concavity changes to the period of the observed beating for the equation of the interval. We concluded that timbre affects dissonance, and it appears that "softer" timbres create less dissonance than "harder" timbres. Our results suggest that the number of concavity changes over the same time interval, the number of concavity changes in one beat period, and the ratio of concavity changes to beat period may be correlated to timbre.

METHODS

Creating equations for component pure tones and complex tones

Table 4

Frequency (Hz)	Sound Pressure Level (dBFS)	Amplitude	Frequency (trigonometric)	Component Equation	Complex Tone Equation
2.58E+02	-2.90E+01	3.55E-04	1.62E+03	y=0.000355sin(1620t)	y=0.000355sin(1620t) +
5.19E+02	-3.78E+01	1.29E-04	3.26E+03	y=0.000129sin(3260t)	0.000129sin(3260t) +
7.79E+02	-4.76E+01	4.17E-05	4.89E+03	y=0.0000417sin(4890t)	0.0000417sin(4890t) + …

Table 4. Shortened table of raw and manipulated data for C4 on piano. The full table contains data for thirteen frequencies, the first thirteen "peaks" in the FFT graph. Following the same structure, tables were created for the nine other piano notes used to create the nine intervals examined in this study.

We used a Fast Fourier Transform (FFT) application (**Figure 1**) on a smartphone (6) to record the notes from C4 (frequency ≈ 261.626 Hz (8)) to C5 (frequency ≈ 523.251 Hz (8), one octave higher than C4) on the piano. The frequencies and amplitudes for the first thirteen "peaks" in the graph (**Figure 1**) were recorded in tables to facilitate the creation of sine equations to represent the pure-tone sound waves at these frequencies. These peaks correspond to the fundamental frequency and twelve harmonics. Some imprecision is introduced because the application rounds the frequencies to integers and the loudness, or sound pressure levels, to one decimal point. **Table 4**, created in Google Sheets, displays the data recorded from the FFT graph (**Figure 1**) as well as amplitude and "trigonometric" frequency values that were calculated. To calculate the amplitude values, the equation

$$L = 20.\,log(\frac{A}{A_0})$$

was used, where L is the loudness in dBFS, A is the amplitude, and A_0 is the reference amplitude which, for the purposes of this study, is arbitrarily assigned the value 0.01. The study is looking at the amplitudes proportional to one another, and the proportions will not be affected by the value of A_0. The value 0.01 was chosen after testing an online graphing program (7) and determining that, since the frequency values are large, zooming in would be required to view individual cycles of the sine waves, and so having small amplitudes for the equations would make the graphs easier to look at. Algebraically, assigning a smaller value to A_0 causes A to have a smaller value. "*log*" denotes the common logarithm in base 10. The equation was manipulated to solve for A (each line of working is separated by a semi-colon).

$$L = 20 \cdot log(\frac{A}{A_0});\ \frac{L}{20} = log(\frac{A}{.01});\ 10^{\frac{L}{20}} = \frac{A}{.01};\ .01(10^{\frac{L}{20}}) = A$$

The function tool in Google Sheets was used to calculate the amplitudes for each frequency. Since the frequencies from the FFT application were given in Hz, they were converted to "trigonometric" frequencies, which represent the number of cycles in an interval of 2π rather than 1. This way, the value of t, the independent variable of the sine equations to be created, equals the number of 1/2π seconds elapsed rather than equaling the number of seconds elapsed. An example of one conversion is below:

$$\frac{258\ cycles}{1\ s} = \frac{x}{2\pi\ s}; x = 516\pi\ cycles; frequency = 516\pi$$

The conversion of all frequency values was done with Google Sheets, yielding the values in the table. The frequencies listed in the table are not in terms of π but have already been multiplied by π to fourteen decimal places.

The sine equations representing pure tones were also created in Google Sheets. Then, the pure tone component equations were added to create an equation representing the resultant complex tone. Physically, when waves travel in the same position or space, they interfere with one another, so that the effects of the waves on the particles they travel through are compounded. This is known as superposition and can be represented mathematically by addition. For example, consider a wave displacing a particle p 1 mm to the right. At the same instant, another wave is displacing particle p 0.5 mm to the right. The effects of these two waves are added, so that the resulting displacement of particle p is 1.5 mm to the right.

Creating graphs of musical intervals

The graphs for the nine musical intervals from a minor second to an octave (**Figure 2**) were produced by graphing the sums of the complex tone equations for C4 and a second note (For example, a perfect fifth is C4 and G4). The horizontal axis of the graphs shows time elapsed in seconds, while the vertical axis shows the sound pressure, or the additional pressure that a sound wave produces in propagating through a medium (9). To obtain the actual sound pressure values, each amplitude value must be multiplied by the actual reference amplitude (A_0) and then by 100 to account for the arbitrarily assigned value of 0.01 to A_0. Because they would all be multiplied by the same factor, their proportions to each other would stay the same. The curves would also have to be shifted upwards so that there are no negative values for sound pressure.

Control experiment

As a control experiment to determine the accuracy of the equations representing intervals produced by adding complex tone equations for individual notes, we recorded a C4 and C#4 played simultaneously on the piano with the FFT application and created an equation following the method described above. The first fourteen rather than thirteen "peaks" were recorded; since two notes are being played, there are two sets of a fundamental frequency and harmonics, so an even number is more suitable. "Peaks" could not be clearly identified visually beyond the first fourteen, so we did not record more "peaks", though this would have increased the accuracy of the resulting equation created. The equation was graphed with the same online graphing program (7) (**Figure 3**).

Finding the number of beat frequencies between 10 and 60 Hz in a musical interval

To obtain the data in Table 1 of the number of beat frequencies between 10 and 60 Hz in a musical interval, we used Google Sheets to calculate the beat frequencies produced by all combinations of component frequencies in the two notes of an interval for all nine intervals examined. The equation for calculating beat frequency is

$$beat\ frequency = |frequency_1 - frequency_2|$$

We highlighted and counted the beat frequencies which were between 10 and 60 Hz and recorded the values.

Finding beat frequencies and periods for the graphs representative of musical intervals

To find approximate beat frequencies and periods for the graphs representative of musical intervals (**Table 1**), we visually estimated the number of full cycles over a certain time interval by examining the graphs in the online graphing program. If cycles were unclear, we adjusted the time interval until cycles became clear. We then calculated the beat frequencies in Hz with the equation

$$beat\ frequency = \frac{number\ of\ cycles}{time\ in\ s}$$

in Google Sheets. The beat periods in seconds were calculated by taking the reciprocals of the beat frequencies

$$beat\ period = \frac{1}{beat\ frequency\ in\ Hz}$$

Finding the number of concavity changes in a graph representative of a musical interval and calculating the ratio of concavity changes to beat period

We calculated equations for the second derivatives of the equations representing musical intervals in Google Sheets using derivative rules. We then graphed these equations in the same online graphing program (7). We counted and recorded the number of horizontal axis crossings over $0 \le t \le 0.3$ s for each graph. The number of horizontal axis crossings in a second derivative graph equals the number of concavity changes in the original graph representative of a musical interval, since the value of the second derivative at any point is the concavity of the original function at that point. We also counted and recorded the number of horizontal axis crossings over one beat period ($0 \le t \le$ beat period) for all musical intervals. We calculated a ratio of concavity changes to beat period for each musical interval with the equation

$$ratio = \frac{number\ of\ concavity\ changes\ in\ one\ beat\ period}{beat\ period}$$

These data are recorded in **Table 1**.

The data processing (excluding plotting) described above was also carried out for the equation from the control experiment.

Comparing different timbres

To address the second part of the aim, "whether timbre has an effect on dissonance", we recorded a human voice singing a C4 and C#4. The same procedure as above was followed. Video editing software (10) was used to overlay audio recordings of the C4 and C#4 notes so that they could be heard sounding simultaneously. The curves for C4 and C#4 by voice were added and the resulting curve was compared to the graph of the C4 and C#4 resultant for piano (**Figure 4**).

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